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A DEVELOPMENT OF ASSOCIATIVE ALGEBRA AND AN ALGEBRAIC THEORY OF NUMBERS, I

H. S. Vandiver

INTRODUCTION: In a number of articles we hope to publish under the above title, it is planned to treat the topics mentioned in a bit unusual way. In the first place, what we shall mean here by the algebraic theory of numbers is the treatment of number theory by means of the methods of abstract algebra rather than by the methods of analysis and geometry, although at times some of the latter may be employed. On the other hand, by the theory of algebraic numbers we shall mean the classical theory of numbers based on the arithmetical properties of the zeroes of a polynomial $f(x)$ with rational integral coefficients and the generalizations of this theory.

In future papers, we shall show that abstract algebra may be applied to even some of the most elementary parts of number theory to obtain results which appear new. An instance of this is well exemplified, using semi-groups, in an article by M. W. Weaver (This magazine, "Co-Sets in a Semi-Group", Vol. 25, pp. 125-36, (1952)). On the other hand, we shall generalize patterns well known in number theory so that it is possible to obtain new developments in abstract algebra.

1.

THE NATURE OF OUR POSTULATES

In the account in the present paper we aim to start close to the beginning of things by setting up a system of postulates for the introduction of associative algebra which postulates are different from those usually given. Here we have in mind, among other things, the fact that, as far as I have been able to find out, many secondary school students are alienated from arithmetic and algebra because the only way they learn these topics in that period is by following a set of rules which are never stated explicitly by the teacher; and the only way the student ultimately is able to carry on the algebraic manipulations correctly is due to the fact that he has heard so many times from his teacher that certain manipulations are wrong. This does not matter much in the case of a student who would never be interested in mathematics, in itself, under any circumstances; but it is rough, it seems to me[1], on the student of innate mathematical ability.

It may be argued that we should be content in learning elementary algebra to become acquainted with the manipulations involved in it just the same way we learn, when children in elementary schools, the use of language. Most of us learn language with little or no knowledge of grammar. I find no fault with this view point concerning the learning of language except that it might happen that a few of the students would have a natural flair for such things as philology. In this case his latent abilities in this direction might not be developed at all until he would reach college. That is, it seems to me, he would be at about the same degree of disadvantage as a young person of natural mathematical ability. It might be best for the teacher to introduce occasionally a few explicit postulates in arithmetic and algebra for the benefit of the more gifted youngsters.

We can now indicate our reasons for employing the types of axioms we introduce here. Consider such an elementary problem as reducing to its simplest form the expression

$$(A) \qquad 6x - (2x + (2x + (x + 2x) + (2x + (x + 1)))).$$

One of the usual procedures would be to say that this expression equals what is obtained from it by substituting $x + 1$ for $(x + 1)$, then to substitute $3x + 1$ for $(2x + x + 1)$, etc., until we reached the stage where we have $6x$ minus an expression contained in a parenthesis and the inside expression contains no parenthesis. Then our problem is a little more complicated. We cannot remove the parenthesis by a substitution immediately in the same manner as before, but the student learns how to handle this minus sign in front of the parenthesis after probably many trials and errors on his part. After learning such rules and ideas in secondary schools, he goes to college and is possibly introduced to a textbook in which perhaps near the beginning it is stated, using symbols, that the postulates governing algebra are as follows: The Commutative Law of Addition; the Closure Law of Addition; the Associative Law of Addition for three elements; if equal numbers be added to equal numbers, their sums be equal; and similar laws governing multiplication; and the Distributive Law. It seems to me that this cannot appear to him except as something entirely new; and in my opinion, it is quite a far cry, using such postulates, to justify the different types of substitution employed in the specific example

[1]In my own case I recall that the only thing that interested me when I was taught arithmetic was the rule for finding the greatest common divisor of two positive integers, which happened when I was about eleven years old. Later, I think, I was first attracted to geometry possibly because some reasons were given for our steps in setting up proofs. This was in spite of the fact that I had very little ability in geometry. It was only in my second year in high school that I became interested in algebra.

just treated; and even if this were done, the student, at the stage mentioned, could follow little, if any, of the necessary arguments. *Our point of view here is exactly the opposite. We start with a set of postulates (1 through 6, including a powerful postulate of substitution) such as the student learns them in elementary schools, and we derive from these postulates not only all the laws just mentioned but others also. In this way we develop elementary algebra, and at the same time we find that we have developed an infinity of finite algebras.* In order to carry out these ideas, we generalize familiar notions and specialize other notions. For example, we do not start off with an abstract notion of a general set as used by most mathematicians. We start with ideas found in common experience and speak of a set of marks or symbols. Having stated certain postulates and developed abstract systems based on them, we later consider modifying these properties so that we cannot regard our "sets" as sets of symbols as, for example, in the theory of real numbers. On the other hand, we generalize ordinary notions of equality. The statement $a = b$ is often taken to stand for a is b. Probably the substitutions we went through in connection with our elementary problem referred to may be justified on such a basis. However, from our point of view $4 + 2 = 6$ does not mean $4 + 2$ is 6. $4 + 2$ will be regarded primarily as a finite ordered set of symbols and as such would be different from 6. So in this way our equality sign seems to have more general significance than usual; in fact, we could conveniently use another symbol, such as $\cong$, for it.

To state conveniently the kind of postulates we need to carry out the ideas before mentioned, we shall in part 3, Foundations of a Theory of the Natural Numbers and Certain Finite Arithmetics, using as far as possible language that we hope will be intelligible even to a non-mathematician, define finite ordered sets of symbols that we shall call "combinations," (cf. the definitions following (3) and footnote 9); thus, (A) is a combination. We do not, however, attempt to describe all the possible methods we shall employ in selecting symbols to denote other types of symbols or sets of symbols, so that our description of procedures with symbols is for that reason incomplete, if not for other reasons.

Another peculiarity of the theory is the fact that we do not say such an expression as

$$a + b + c$$

is an abbreviation for

$$((a + b) + c).$$

(This is only possible since we are confining ourselves to associative systems). If we did, then from the point of view we are using, this would make things very difficult and complicated for us. We would not only have to define such an expression as the second one as being a combination, but also the first one as a finite ordered set of symbols which was written as an abbreviation of the second. We shall attempt to define "combination" in such a way that each of the last expressions will be recognized as such.

2.

THE NATURAL NUMBERS, DENUMERABLE SETS, AND SYSTEMS OF SINGLE COMPOSITION

Before discussing the foundations of algebra, a system of double composition, we introduce a system of single composition which from our standpoint is much simpler. The natural numbers are defined, in this section, but symbols such as + and × are not employed in connection with them.

We start with the notion of a set of symbols or marks. We shall also refer to them as elements. We then consider some of the properties of said symbols that we observe from ordinary experience. As we bring up the idea of each symbol in succession in our mind, we shall refer to an immediate predecessor of a symbol and an immediate successor of it; the first symbol (with no immediate predecessor) and the last symbol (with no immediate successor). We illustrate this mental process by means of

(I) $$a,\ b,\ c,\ d,\ e,\ f$$

in which we refer to a as the first symbol or element in the set; f is the last element; d is the immediate predecessor of e in the set; and c is the immediate successor of b in the set. Writing the symbols in this manner is quite suggestive of the order of thought we have employed. If we had illustrated our process by means of a circular order, correspondence is not so clear. In

(II)
$$\begin{array}{ccc} & a & \\ f & & b \\ e & & c \\ & d & \end{array}$$

we would immediately have to define the first term or the first element, as a, and f as the last element; otherwise, we would have to consider the repetition of elements. This leads to the idea of *different* symbols and the same symbol. In view of these ideas, we shall refer to a set of symbols such as

(III) $$r,\ s,\ r,\ t,\ v,\ s$$

as a set "with repetitions." This we can only illustrate by saying that a and b are different symbols and a is the same symbol as a.[2] Hence, our writing the symbols on a line suggests all the concepts we have talked of, and we will call such a set a set of linearly ordered symbols. We refer to (I) or (III) as a finite (linearly) ordered set of symbols. Such a set is non-null and the first term may also be the last. Referring to any set of symbols, we shall speak of replacing a symbol in it by another or by itself, or of selecting certain symbols from a set of symbols. If a set S is such that by taking any symbol contained in it and comparing with any other symbol in it (assuming it includes another), then said symbols are different, S is said to be *a set of different symbols, or a set with no repetitions.* If we have a set S' with an element, say a, and no other, then S' is also said to be a set with no repetitions.

To introduce the ideas of correspondence and counting, we extend the notion of ordered sets of symbols by indicating the familiar idea in ordinary experience of a set of sets of symbols. Thus, we can speak of $a;\ b;\ c,\ d,\ e$ as a finite linearly ordered set of sets of symbols in which the set b is the immediate successor of the set a; c, d, e is the immediate successor of the set b, with c, d, e being the last set in this set of sets. We consider the symbols

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,$$

known as digits, and the finite linearly ordered set

$$\text{(IV)} \qquad 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9.$$

The last element in this set is 9, but we may, without using any explicit notion of addition, change the status of this element by introducing an immediate successor of 9. To do this we employ the digit 0 and take as the immediate successor of the set 9 the set 10; as the immediate successor of 10, we take 11, and so on, in line with the usual decimal representation (not defined here). We shall denote this extended set by $N.N.$ From this standpoint we shall refer to any of the finite ordered sets in $N.N.$ as a natural number, and we will refer to the whole set as the set of natural numbers. The natural numbers satisfy the following postulates:

Postulate I. 1 is a natural number.

Postulate II. Each natural number has an immediate successor in $N.N.$

This Postulate II transcends our ordinary experience in connection with the symbols in the sense that the set of natural numbers has no last term; and here we seem to be introducing a definitely mathematical

[2]Note that we are not introducing any idea as yet of the equality or inequality of symbols. From our standpoint this will be a much more general idea. We use the term "different" here, in place of "distinct," as the latter will be used later to indicate unequal symbols.

idea.

As is usual in algebraic discussions, we shall use the idea of "denoting." We use a symbol to denote a set of symbols, or, in particular, a symbol itself; and in any statement made or relation discussed here, we may replace any set of symbols by some letter denoting it and vice versa. In particular, a symbol may denote itself. However, as stated in our introduction, we do not attempt to set up a complete set of rules governing denoting.

We assume we have a set of symbols S. We shall now speak of affixing subscripts to symbols, which latter we have already selected; thus, we obtain a_i from a by affixing a subscript i. Next we will speak of affixing an additional subscript, say j, such that we obtain a_{ij}, etc. Suppose we have a set of symbols and select an element therein. To this element we affix the subscript 1, select another element and affix the subscript 2, etc., until we have reached a natural number denoted by n as a subscript. Further, suppose there is no element of our original set onto which we have not affixed a subscript. We then say that our original set S is finite and contains n elements. Taking any symbol, such as a, we shall say that we may denote the elements of S by a_1, a_2, ..., a_n. If we have a linearly ordered set of symbols with no last term, such as the set of natural numbers, we shall call such a set infinite.[3] *We now assume that all the sets discussed here have the property that the elements of any one may be denoted by* a_1, a_2, a_3, ..., *where the subscripts range over the set* (*N.N.*) *or over the set* 1, 2, 3, ..., n, *where* n *denotes some natural number. In other words, the sets considered by us are denumerable.*[3]

Sub-sets. Consider a set S_1 consisting of the elements a_i, where i ranges over the set 1, 2, ..., n, n denoting a natural number, or over the set of natural numbers (*N.N.*). Suppose it is possible to select from S_1 in some fashion a set S_2 consisting of different symbols, then S_2 is said to be a *sub-set* of S_1. If there is a natural number denoted by k such that a_k belongs to S_1, but not S_2, then S_2 is called a *proper sub-set* of S_1. Since the elements of any denumerable set can be denoted by a set of a's as above defined, then the definitions above apply to any denumerable set of symbols.

[3]It seems that there are two different notions about enumerating the elements of sets with repetitions. Thus, it would appear that some writers would say the set (III) contains four elements, but six elements "counting repetitions." If we follow out our definition, however, in counting the elements of (III), we would obtain a set r_1, s_2, r_3, t_4, v_5, s_6, which also by our definition contains six elements. However, it is apparently consistent with our previous definitions to say that (III) contains four different elements. If the reader thinks that we are unduly preoccupied with sets having repetitions, we wish to point out that most of the sets we talk of in this article contain them. Thus, from our standpoint $((a + b) + c) + d$ is merely a finite ordered set of symbols; and we would be in a bad way, as far as our theory is concerned, if we could not say that this expression contains two parentheses.

Suppose we have a set of symbols S_1, whose elements are denoted by $a_1, a_2, \ldots, a_m$, and another set S_2, whose elements are denoted by $b_1, b_2, \ldots, b_n$. A *correspondence* or *mapping* is a rule which determines for each element a_i of S_1 an element b_j of S_2 which is said to correspond to a_i and such that no different symbol in S_2 corresponds to a_i. Conversely, each element of S_2 is the correspondent of at least one symbol in S_1. In particular, the correspondence is said to be one-one when a_i corresponds to b_j but to no different symbol in S_2, and b_h corresponds to a_k in S_1 and no different symbol in S_1. For example, if we have the set a_{11}, a_{12}, a_{21}, a_{22}, and b_1 and b_2, we can indicate what we call a two-one correspondence which may be written symbolically as

$$\left.\begin{matrix} a_{11} \\ a_{12} \end{matrix}\right> b_1, \qquad \left.\begin{matrix} a_{21} \\ a_{22} \end{matrix}\right> b_2.$$

If we have a set a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4, then we may indicate a certain one-one correspondence between these two sets by

$$a_i \longleftrightarrow b_i,$$

i ranging over 1, 2, 3, 4. Also, the first set is said to be mapped on the second and conversely. We employ a similar idea in connection with infinite sets.

In referring to an ordered set, we shall speak of a symbol contained in it and then obtaining a type of sub-set by selecting the symbols following this in order as they appear in the original set. Thus, we may select the element c in (I) and obtain a set by taking the immediate successor of c in (I), namely d, and the immediate successor of d, namely e, and obtain, if we wish to use e as the last element of one set, the set c, d, e. But d, e, a is not a set of this type.

We now introduce and discuss rather informally a system of single composition. Consider a finite set of symbols without repetitions which we shall call S. Suppose S contains n elements. Consider another set S_1 containing n elements which is obtained from S by selecting elements in the first set. If S_1 contains no repetitions, we call S_1 a permutation of S and vice versa. Thus $bcad$ is a permutation of $abcd$. We now introduce a symbol

$$\begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}.$$

In order to set up a system of operation with such symbols, we first note that we can consider the columns which appear in it. We will state that the first symbol is equal to any symbol obtained by interchanging the columns; thus

$$\begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} = \begin{pmatrix} c & b & a & d \\ a & c & b & d \end{pmatrix}.$$

More generally we set

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a'_1 & a'_2 & \dots & a'_n \end{pmatrix} = \begin{pmatrix} a_{i_1} & a_{i_2} & \dots & a_{i_n} \\ a'_{i_1} & a'_{i_2} & \dots & a'_{i_n} \end{pmatrix},$$

where each row in each symbol is a permutation of the original elements $a_1, a_2, \dots, a_n$; and if $a_{i_t} = a_j$, $t = 1, 2, \dots, n$, then $a'_{i_t} = a'_j$. Let

$$S_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a'_1 & a'_2 & \dots & a'_n \end{pmatrix}, \quad S_2 = \begin{pmatrix} a'_1 & a'_2 & \dots & a'_n \\ a''_1 & a''_2 & \dots & a''_n \end{pmatrix}, \quad T = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a''_1 & a''_2 & \dots & a''_n \end{pmatrix}.$$

We call T the product of S_1 and S_2 and write

$$S_1 S_2 = T.$$

In the above, as before, each row in each symbol is a permutation of $a_1, a_2, \dots, a_n$. Concerning the equality symbol, =, it seems reasonable to postulate then that if $S_1 = S_2$, then $S_2 = S_1$; and also $S_1 = S_1$. We shall call these symbols, such as S_1, that we have been using, substitutions. We note, in particular, there is a substitution

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

This substitution is called the identity substitution, and we shall denote it by I. We note also that if we multiply S_1 by

$$\begin{pmatrix} a'_1 & a'_2 & \dots & a'_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix},$$

we obtain I. The second of these substitutions is called inverse of the first, and it is usually denoted by S_1^{-1}. We note also that

(1) $$S_1^{-1} S_1 = I.$$

We write down what we call a product of three substitutions as $S_1 S_2 S_3$. We can interpret it as the substitution obtained by taking the product $S_1 S_2$ and taking this resulting product with S_3. This gives us the single substitution T_1, and we write $S_1 S_2 S_3 = T_1$. Using our definitions, we find that if we take the product of S_1 times the single substitution which equals $S_2 S_3$ that T_1 is also obtained; and we write this result as

$$(S_1S_2)S_3 = S_1(S_2S_3).$$

This is called the associative law for three substitutions. In view of (1), we find that corresponding to any given substitutions S and T, there is a substitution S_x such that

$$SS_x = T, \tag{2}$$

and an S_y such that

$$S_yS = T. \tag{2a}$$

In view of this property, the associative law, and the fact that the product of any two substitutions is a substitution as well as two substitutions being equal or unequal, these conditions being mutually exclusive, the set of substitutions on n letters is said to be a group. If the properties (2) and (2a) do not necessarily both hold in a set of elements which have the other properties mentioned, such a set is called a semi-group. The formal definition of a semi-group and group in general will appear in a later paper.

3.

FOUNDATIONS OF A THEORY OF THE NATURAL NUMBERS AND CERTAIN FINITE ARITHMETICS

In this part we shall develop a set of postulates concerning certain symbols in a system, which system will include as special cases not only elementary arithmetic but various types of finite arithmetics.[4] One of the principal reasons we develop these systems simultaneously is to justify the use of the word *existence* in connection with the use of systems of double composition which cannot be embedded in any ring[5]

[4]The theory which I shall consider here was developed mainly during seminars on abstract algebra and number theory which I gave at the University of Texas during the last 20 years. During this period, I also discussed some of the ideas in personal conversations with various mathematicians. As a result of this, I am indebted for suggestions and corrections to F. C. Biesele, A. Church, J. L. Dorroh, O. B. Faircloth, H. C. Miller, J. B. Rosser, J. M. Slye, W. J. Viavant, and M. W. Weaver. However, none of these individuals should be held responsible for any errors or obscurities which appear in this paper since the decision to write it in the form in which it now appears was entirely my own.

[5]These ideas were discussed in part in the following papers by the writer:

1. "On the Foundations of a Constructive Theory of Discrete and Commutative Algebra," *Proc. Nat'l. Acad. Sci.*, Vol. 20, 579-584, 1934.
2. "Note on a Simple Type of Algebra in which the Cancellation Law of Addition does not hold," *Bull. Amer. Math. Soc.*, Vol. 40, 914-920, 1934.
3. "On the Foundations of a Constructive Theory of Discrete Commutative

(the terms ring and embedded being employed here as usually defined).

We now consider a set of symbols

(3) $$C_1,\ C_2,\ C_3,\ \ldots\ldots\ldots$$

Denote some natural number by k. Then in (3) we define the immediate successor of C_k as $C_{k'}$, where k' is the immediate successor of k in N.N. We shall now introduce in addition to these symbols a symbol + (called a plus sign), and × (called a multiplication sign), and (, called a left parenthesis symbol, and), called a right parenthesis symbol. We shall define the term *combination*[6] in connection with the symbols.

Definition. Any of the symbols in (3) or any symbol denoting any of them is said to be a *combination.* If A denotes a combination and B also, then $A + B$ is said to be a *combination,* also A, (A) and $A \times B$. A *sub-combination* of a combination A is a combination consisting of a symbol contained in A or else such a symbol followed by others in order as they appear in A.

Definition. If A denotes a combination, then (A) is called a *parenthesis enclosed combination.*

Definition. A *closed combination* C is a combination such that if any + sign occurs in it, there is a sub-combination of C which contains this + sign, and which is also a parenthesis enclosed combination. If a combination contains no plus sign it is said to be closed.

We introduce a symbol of relation with the combinations, =, called equality. In the following statements, each capital letter or capital letter primed denotes an arbitrary combination as above described or a combination limited in character by the conditions in the statements. A small letter denotes an arbitrary natural number, or else a natural

Algebra," *Proc. Nat'l. Acad. Sci.*, vol. 21, 162-165, 1935.

4. "On some Simple Types of Semi-Rings," *Amer. Math. Monthly*, vol. **46**, 22-26, 1939.

The idea of using the postulate of substitution as employed here was defined by the author for semi-rings in his reference last mentioned, p. 26, and for semi-groups in "The Elements of a Theory of Abstract Discrete Semi-Groups," *Vierteljahrschrift Natur. Gesell.*, Zurich, v. 46, 121-123, 1940. It was also used by Stephen A. Kiss in his book, "Transformations on Lattices and Structure of Logic," New York, 1947, for semi-groups and more general systems, and as a concept of logic, as applied to arithmetic by Birkhoff and MacLane, "Survey of Modern Algebra," pp. 30-31, New York, 1941. From the point of view we use here, the substitution postulate is a little complicated since we are trying to take strict account of all the parentheses that appear in systems of double composition. I imagine that this substitution principle has been stated by a number of other authors, but I have not yet noted where.

[6]It does not seem to be the usual thing among writers along these lines to define combination as we do here. The idea seems to be that if the closure law holds, then we can always replace a combination by an element of our set which it is equal to. From the theory we are using, however, this does not seem possible, as we do not assume the closure law but prove it (Theorems 10 and 11) by means of our postulates.

number limited in character by the conditions in the statement. An equality is also called a statement.

Postulate 1. (Identity) $A = A$.

Postulate 2. (Parenthesis) $(A) = A$.

Postulate 3. (Substitution) If $A = B$ and $D = C$, where C denotes a sub-combination of B and B' denotes the combination obtained from B by putting D in place of C, then $B' = A$, provided that if C is immediately preceeded by or immediately succeeded by an $\times$ sign in B, then C and also D must be closed combinations.

Postulate 4. (Induction) For each natural number n let there be associated a statement denoted by $S(n)$. If $S(1)$ holds and if it follows that if $S(a)$ holds, then $S(a')$ holds, where a' is the immediate successor of a in the set of natural numbers, then $S(n)$ holds for each natural number n.

Postulate 5. (Addition) If n denotes a natural number and n' denotes the immediate successor to this number in the set of natural numbers, then

$$C_n + C_1 = C_{n'}.$$

Postulate 6. (Multiplication)

$$C_a \times (C_b + C_1) = C_a \times C_b + C_a.$$

$$C_a \times C_1 = C_a$$

We may then prove the following (proofs omitted except for Theorems 6 & 7):[7]

Theorem 1. (Symmetry) If $A = B$, then $B = A$.

Theorem 2. (Transitivity) If $A = B$ and $B = C$, then $A = C$.

Theorem 3. (Composition under addition) If $A = B$ and $C = D$, then $A + C = B + D$.

Theorem 4. (Composition under multiplication) If $A = B$ and $C = D$, and if each letter denotes a closed combination, then $A \times C = B \times D$.

Theorem 5. (General Substitution) If $E = F$ and $G = H$, where G denotes a sub-combination of E, and E' denotes the combination obtained

[7]We may employ as postulates our present Postulates 1, 2, 4, 5, and 6, and also employ our present Theorems 1, 2, 3, and 4 in liew of the Postulates 1-6, which we have used here. It may be shown that the two sets are equivalent.

It may be noted that it is often very difficult to verify that a given system has the property stated in Postulate 3. To indicate this we might consider the introduction of the negative integers by means of ordered pairs by the usual method. Verification that combinations of these ordered pairs of natural numbers satisfy Postulate 3 would appear difficult, if not impossible. However, we shall show elsewhere that using a certain type of isomorphism, as applied to the natural numbers only, we may adjoin zero, the negative numbers, and the rational fractions.

from E by putting H in place of G, then $E' = F$ provided that if G is immediately preceded by or immediately succeeded by a $\times$ sign in E, then G and H must be closed combinations. Similarly, if G is a sub-combination of F and F' is obtained from F by putting H in place of G, then $E = F'$ with the above mentioned restrictions on G and H.

Theorem 6. (Commutative law of addition)

$$C_a + C_b = C_b + C_a. \tag{4}$$

Proof: We first show that

$$C_a + C_1 = C_1 + C_a. \tag{5}$$

We note first that (5) holds for 1 in place of a, since

$$C_1 + C_1 = C_1 + C_1$$

in view of Postulate 1 and the fact that the expressions on each side of the equality are combinations. Hence, the first condition in Postulate 4 is satisfied. Using the second part of Postulate 4, we assume

$$C_k + C_1 = C_1 + C_k;$$

and employing Postulate 1 and Theorem 3, we have

$$C_k + C_1 + C_1 = C_1 + C_k + C_1. \tag{6}$$

We then note that $C_k + C_1$ is a sub-combination of each of the combinations in (6) by definition. Now introduce $C_k + C_1 = C_{k'}$ by Postulate 5. We then make a substitution on each side of (6) by Theorem 5, employing the last relation, and we have (5) with k' in place of a. The relation (4) holds then for 1 in place of b. Assume

$$C_a + C_k = C_k + C_a.$$

The use of Theorem 3 and Postulate 1 gives

$$C_a + C_k + C_1 = C_k + C_a + C_1.$$

In view of (5) and Theorem 5, we obtain

$$C_a + C_k + C_1 = C_k + C_1 + C_a.$$

Whence, by Postulate 5 and Theorem 5, we find

$$C_a + C_{k'} = C_{k'} + C_a$$

which by Postulate 4 gives (4).

Theorem 7. (Associative law of addition).

(7) $$(A + B) + D = A + (B + D).$$

Proof: The proof is quite simple because of the character of our particular set of postulates. Since $A + B + D$, $A + B$, and $B + D$ are combinations, then by Postulate 1 we have

(8) $$A + B + D = A + B + D;$$

and by Postulate 2 we obtain

$$A + B = (A + B), \text{ and } B + D = (B + D).$$

Since both $A + B$ and $B + D$ are sub-combinations of $A + B + D$, we may take (8) and employ two substitutions, using Theorem 5, and obtain (7).

Theorem 8. (Associative law of multiplication).

$$(A \times B) \times D = A \times (B \times D),$$

where A, B, and D denote closed combinations.[8]

Theorem 9.

$$C_1 \times C_m = C_m.$$

Theorem 10. (Closure law of addition). If a and b denote given natural numbers, then we may obtain a C_s such that

$$C_a + C_b = C_s,$$

where s denotes some natural number.

Theorem 11. (Closure law of multiplication). With a and b defined as in Theorem 10, we may obtain a C_t such that

$$C_a \times C_b = C_t,$$

where t denotes some natural number.

Theorem 12. (Commutative law of multiplication).

$$C_m \times C_n = C_n \times C_m.$$

Theorem 13. (Distributive law).

[8]An alternative method for handling the symbols denoting combinations would be to modify a bit the property of denoting we have already mentioned, namely, that if any set is denoted by a letter, then we can interchange the set or letter in any statement or relation that we consider. We might say that the last statement was subject to the restriction that if A denotes some combination, then it can be replaced by the combination that it denotes except that if A appears in an equation where it is immediately preceded by or immediately succeeded by x, then the combination in question must be a closed combination. Using the modification just stated, we could have omitted the exceptions in the statement of several of the theorems stated above.

$$C_a \times (C_b + C_k) = C_a \times C_b + C_a \times C_k$$

and

$$(C_b + C_n) \times C_a = C_b \times C_a + C_n \times C_a.$$

We will now discuss informally several types of algebras obtained from the C's.

So far nothing has been said concerning inequality or equality among the C's. Let us assume that each element in the set (3) is equal to C_1, then our algebra of the C's consists of one distinct element, namely C_1. We have from our postulates for the C's the following:

$$C_1 + C_1 = C_1$$

and

$$C_1 \times C_1 = C_1.$$

Now instead of the algebra we just described, suppose we take the set (3) and assume that for some natural number, not 1, and denoted by m, we have, if m' is the immediate successor of m,

(9) $$C_{m'} = C_1$$

but that

$$C_1 \neq C_k$$

if k denotes a natural number in the set

$$2,\ 3,\ \ldots\ldots\ldots,\ m.$$

The symbol $\neq$ reads "is unequal to" and is here introduced in this article for the first time. We also assume that either $C_a = C_b$ or $C_a \neq C_b$, for a and b denoting any natural numbers, and that only one of these two relations holds. Since, from the above, $C_1 + C_m = C_1$, we obtain immediately by induction on a

$$C_a + C_m = C_a,$$

that is, C_m acts as a zero element, as defined in elementary algebra, in our set under addition. We also have from Theorem 9

$$C_1 \times C_m = C_m,$$

and we easily obtain

$$C_a \times C_m = C_m \times C_a = C_m$$

by induction on a. This shows that C_m also acts like a zero element under multiplication.

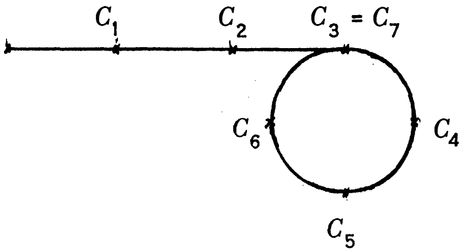
We shall now consider a second type of finite algebra. Again refer to the set (3) and assume that

$$C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6$$

are unequal but that $C_7 = C_3$; then it follows from Postulate 5

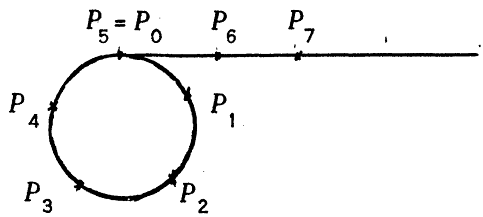
$$C_6 + C_1 = C_2 + C_1.$$

Yet, we cannot cancel the C_1's if we use the assumption already referred to that any two C's cannot be equal and unequal at the same time. The algebra just described is illustrated by the following figure:



Let C_a indicate the operation of passing over a equal units of distance in this figure and assume that $C_a = C_b$ if, and only if, the operation designated by C_a brings us to the same point as the operation designated by C_b. Then it is clear that $C_7 = C_3$, and starting with C_7 the elements in (3) repeat in cycles, the elements in each cycle equaling C_3, C_4, C_5, and C_6 in order. There is no element having the property of the zero element in this algebra. Also, the cancellation law of multiplication does not hold in general. Further, what corresponds to division is not always possible. The latter two remarks apply also to the algebra previously defined in connection with (9).

An algebra of a quite different character than any discussed so far in this paper may be obtained by considering the following geometric figure:



where the line at the top extends indefinitely to the right. Here we have $P_0 = P_5$; yet,

$$P_0 + P_1 \neq P_5 + P_1,$$

that is, the composition law of addition does not hold. On the other hand, it will be found that the cancellation law of addition holds; and it is clear that the associative and commutative laws hold. This brings out how important our substitution postulate is.

Development of the Arithmetic of the Natural Numbers: We now consider the set (3) again, and *resume our formal discussion.* Assume that $C_a = C_b$ if, and only if, a and b denote the same natural number. On the other hand, if a and b do not denote the same natural number, we write $C_a \neq C_b$. That is, we shall say that the elements in (3) will now be assumed distinct. Also, we will now replace in any of the relations, we have so far obtained, involving the C's C_k by k so that corresponding to

(10) $$C_a + C_b = C_d$$

and

(11) $$C_a \times C_b = C_e,$$

we have

(12) $$a + b = d$$

and

(13) $$a \times b = e,$$

and conversely.

The set (3) and the set of natural numbers as now used are said to be *isomorphic* since there is a one-one (biunique) correspondence between C_a and a, which is written

$$C_a \leftrightarrow a,$$

and the relations (10), (11), (12), and (13) hold. From now on we omit the symbol $\times$, and write $a \times b$ as ab. We carry over the definition of combinations of the C's to corresponding expressions with natural numbers replacing the C's. Then we introduce the

Postulate 7. If a and b denote natural numbers, and if

$$a + k = b,$$

where k denotes a natural number, then we write

$$b > a, \text{ and } a < b.$$

Also if either $b > a$ or $a < b$, then we may obtain an s such that

$$a + s = b,$$

where s denotes a natural number. If e and f denote natural numbers, neither = 1, then the statements

$$e = f, \ e > f, \text{ and } e < f$$

are mutually exclusive, that is, one and just one of these relations holds. Also, one and just one of the relations $g = 1$, $g > 1$ holds.

If a and b denote natural numbers, and if we introduce one of the three statements

$$a > b, \ a < b, \text{ and } a = b \tag{14}$$

and employ it in connection with our postulates and established theorems, and derive any two of the relations

$$d = e, \ d > e, \ d < e,$$

where d and e denote natural numbers, this is said to be a contradiction; and the particular one of the three relations (14) which we introduced is said to be false.

We have, if all letters denote natural numbers,

Theorem 14. If $a > b$ and $b > c$, then $a > c$.

Theorem 15. If $a > b$ and $c > d$, then $a + c > b + d$.

Theorem 16. If $a > b$ and $c > d$, then $ac > bd$.

Theorem 17. If $a + c = b + c$, then $a = b$.

Theorem 18. If $ac = bc$, then $a = b$.

In our next article under the present title, we hope to discuss the concepts[9] and some of the properties of semi-groups and semi-rings, and adjoin zero and the negative integers to the set of positive integers.

[9]Concerning the definition of combination given previously, which employs a sort of induction, the combination may be defined directly as follows:

Definition. Consider a finite linearly ordered set of symbols containing only symbols of the following type: symbols (letters) denoting elements in a set of symbols described in (3), symbols of conjunction + and ×, parenthesis symbols (and) which will be called a left parenthesis symbol (abbreviated L.P.S.) and a right parenthesis symbol (abbreviated R.P.S.), respectively, and such that:

1. It contains at least one symbol denoting an element of (3).
2. It begins with either a L.P.S. or a symbol denoting an element of (3) and ends with either a R.P.S. or a symbol denoting an element of (3).
3. It has no L.P.S. immediately preceding a symbol other than another L.P.S. or a symbol denoting an element of (3) and no R.P.S. immediately preceded by a symbol other than an R.P.S. or a symbol denoting an element of (3).
4. Any two successive symbols denoting elements of (3) are separated by just one symbol of conjunction.
5. There exists in it a one-one correspondence between the set of all L.P.S.'s and the set of all R.P.S.'s such that:

(a) To each L.P.S. there corresponds an R.P.S., which follows it, and

(b) If either parenthesis symbol of a given pair lies between the two parentheses of the second pair, then the other parenthesis of the first pair lies between two parentheses of the second pair.

The ordered set just described is said to be a *combination*. Further, if we replace any of the symbols denoting elements in the set of symbols described in (3), which appear in the combination just mentioned, by symbols denoting combinations, the resulting set is also said to be a combination.

The University of Texas

COLLEGIATE ARTICLES

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FOUNDATIONS OF OPERATOR MATHEMATICS

Jerome Hines

Foreword:

The fundamental ideas in the theory of operator mathematics were originally conceived through certain formal procedures occurring in differential equations. Because of the fact that many non-differentiable functions are integrable the modern treatment of operators has shifted from the differential approach to the integral equation approach. Unfortunately, the integral equation approach forces us to abandon many powerful generalizing tools offered by the more direct differential method. One such tool is the operator equation which relates operators without reference to specific operands. Examples of such operators are the ordinary sine, cosine, logarithm, derivative, etc.

The main purpose of this paper is to lay a foundation for the algebra and calculus of operators from the differential approach. The structure of this approach permits us to define operations upon operators. These definitions will arise from similarities between operator and ordinary equations.

1. Operator Equations:

An operator is the representation of a transformation. An operand is that which is transformed. An opus is the result of an operation. We shall denote operators by capital letters and operands and opi by small letters, except when noted in the context.

For simple cases it will suffice to indicate operation by following the operator by the operand with no sign between them. To show the equivalence of Aa and its opus, b, we shall use the equality sign, i.e.

$$Aa = b$$

This equation means that the result of applying the operator, A, to the operand, a, is the opus, b.

If the operand, a, is any function of a complex variable, say, for which Aa and Bb are defined, and

$$Aa = b$$

and

$$Ba = b$$

for all a's, then A is said to be identical to B in the field, S, of such complex variables.

We shall denote the identity of two operators, A and B, for the field, S, by

1.1 $$A \overset{S}{=} B$$

This equation is called an operator equation since its terms involve only symbols on the operator level. Operator equations can be given many properties similar to ordinary equations but they do not always obey the laws of simple algebra.

Definition I:

Given the operators, P and Q, then $P \pm Q$ is an operator defined by the relation

$$(P \pm Q)t \equiv Pt + Qt \text{ (for all } t)$$

Definition II:

Given the operators, P and Q, if $Qt = r$, has meaning, then PQ is an operator defined by the relation

$$PQt = Pr$$

PQ represents successive applications of Q and P, provided Pr has meaning. This may be extended to the successive applications of any number of operators, e.g. $PQRS\,t$ denotes the successive applications of S, R, Q, and P. We add the convention that if a variable or constant appears in the place of an operator in an operator equation with no signs between it and its juxtaposed symbols, it shall imply the operation of multiplication by that variable or constant. For example, nAB implies multiplying the opus of A on b by the variable, n.

Definition III:

We define the upper right hand index of any operator, H, by the properties

1.2 $$H^1 \overset{S}{=} H$$

1.3 $$H^0 \overset{S}{=} I$$

and

1.4 $$H^m H^n \overset{S}{=} H^{m+n}$$

I is generally called the identity operator, where $It \equiv t$. This definition includes negative indices if they have an unique existence. It immediately follows that, for p an integer, H^p represents p-successive applications of the operator, H.

Definition IV:

We define the null operator, Ω, by the relation

$$\Omega t \equiv 0 \quad \text{(for all } t)$$

The following are results of these definitions:

a) If

$$At \equiv Bt \quad (\text{for all } t)$$

then, adding (or subtracting) equal quantities, Et, to both sides of the identity gives us

$$At \pm Et \equiv Bt \pm Et$$

whence, by Definition I,

1.5 $$A \pm E \overset{S}{=} B \pm E$$

Thus we can add (or subtract) identical operators to each side of an operator equation.

b) If

$$A \overset{S}{=} B$$

and At and Bt are of the class, S, then

1.6 $$DA \overset{S}{=} DB$$

where DA and DB mean successive applications of A and D, and B and D. Thus we can operate from the left by identical operators on an operator equation.

c) Again, if $$A \overset{S}{=} B$$

and Et is of the class S, then

1.7 $$AE \overset{S}{=} BE$$

Thus, operation from the right by identical operators on an operator equation is also permissible.

d) If there exists an operator, B, such that

$$BAb \equiv A^{\circ}b$$

$$\equiv b \quad (\text{for all } b)$$

then, by equation 1.1, and Definitions II and III,

1.8 $$B \overset{S}{=} A^{-1}$$

i.e. B is the inverse of A.

e) $(AB)^{-1}$ is an operator such that

$$(AB)^{-1}AB \overset{S}{=} I$$

Provided there exists an unique operator $(AB)^{-1}$ fulfilling the above condition it follows from equation 1.1 and Definition II that

1.9 $$(AB)^{-1} \overset{S}{=} B^{-1}A^{-1}$$

f) From the convention of multipliers in Definition II and the definitions of the operators, I and Ω, it follows that the unit and zero multipliers, 1 and 0, are given by

1.10 $$\Omega \overset{S}{=} 0$$

and

1.11 $$\mathrm{I} \overset{S}{=} 1$$

g) From the above definitions it can also be easily shown that

1.12 $$\mathrm{I}A \overset{S}{=} A$$

and

1.13 $$\Omega A \overset{S}{=} \Omega$$

2. Higher Operator Forms:

In this section we shall speak of the symbolic formation of new operators by the operation of specific operators upon others. Examples of these new operators will be clearly defined in terms of simple operators in this section but first we must extend our conventions. The symbolic operation of the operator, A, upon the operator-operand, B, will be denoted by $A \cdot B$. When $A \cdot B$ is defined, then

2.1 $$A \cdot Bd = e$$

will denote that we apply the operator, $A \cdot B$, to the operand, d, obtaining the opus, e. Generally this will be quite different from successive applications of B and A denoted by

$$ABd = e$$

Further,

$$A \cdot BCd = e$$

will denote that we apply the operator, C, to the operand, d, and then apply the new operator, $A \cdot B$, to its opus to obtain the final opus, e. For higher operator products we can employ two or more dots arranged vertically to indicate intimacy of operation, e.g.

$$F \cdot G : HJb = c$$

denotes that we first apply the operator, J, to the operand, b. Then a new operator is formed by symbolically applying first G to H, then F to $G : H$, and this resulting operator is applied to the opus of J on b, the final opus being c. This has an entirely different meaning than

$$FGHJb = c$$

which only signifies successive applications of J, H, G, and F to b

giving c.

Definition V:

If n is any complex variable, we define

$$n \cdot A \overset{S}{=} nA$$

This is the degenerate case where the symbolic operator $n \cdot A$ is equivalent to the successive applications of A and n. It follows from definition V and equations 1.10 and 1.11 that

2.2 $$I \cdot A \overset{S}{=} A$$

and

2.3 $$\Omega \cdot A \overset{S}{=} \Omega$$

In order to build a function theory between operators in one variable and a corresponding calculus it will be helpful to construct definitions such that ordinary function forms such as log, sin, and cos when treated as operators will be considered as independent of the variable in the operand. Operators that may in this manner be considered independent of the operand-variable will be called functors. Examples of non-functor operators are $\log_e$, *sin,* and *cos*. In the theory of matrices the derivative of an operator, A, is given by the expression

$$D \cdot A \overset{S}{=} DA - AD$$

This equation might be used as the definition for the symbolic "derivative of an operator" except that the derivative of a functor would not be equal to the null operator unless it commuted with the derivative. Hence we would lose a much desired parallelism with ordinary calculus, where the derivative of a constant is zero. A broader definition of the derivative of an operator will be given in the next section which includes the above equation as a special case but includes the condition that the derivative of a functor is the null operator.

The following operator notations are convenient:

2.4 $$L_{(\,)} \overset{S}{=} \operatorname*{Limit}_{(\,)\to 0}$$

2.5 $${}_h\delta_x f(x) \equiv f(x+h) \qquad (\text{for all } f(x))$$

2.6 $${}_h\Delta_x = {}_h\delta_x - I$$

We shall use ${}_h\delta_x$, as defined above, as an operator although δ is not a capital letter.

It follows from Definition I and equations 2.5 and 2.6 that

$${}_h\Delta_x f(x)g(x) \equiv f(x+h)g(x+h) - f(x)g(x)$$

whence from 2.5

2.7 $$_h\Delta_x f(x)g(x) \equiv {_h\delta_x} \cdot [f(x)]_h\delta_x \cdot [g(x)] - f(x)g(x)$$

Bear in mind that the dot indicates intimacy of operation, i.e. $_h\delta_x$ applies only to the function immediately following the dot. $_h\delta_x \cdot [g(x)]$ has meaning since

$$_h\delta_x \cdot [g(x)] \equiv {_h\delta_x} g(x)$$

Thus 2.7 can be written

2.8 $$_h\Delta_x f(x)g(x) \equiv {_h\delta_x} \cdot [f(x)]_h\delta_x g(x) - f(x)g(x)$$

We can consider $f(x)$ as a multiplier. It is then an operator. $_h\Delta_x f(x)$ can likewise be interpreted as an operator. Thus, by 1.6 we can consider $g(x)$ as the operand on the left side of 2.8. Similarly the right hand member of 2.8 contains the operator, $_h\delta_x \cdot [f(x)]_h\delta_x$ and the operand, $g(x)$. Then by 1.1 we can rewrite 2.8 in the form

2.9 $$_h\Delta_x f(x) \overset{S}{=} {_h\delta_x} \cdot [f(x)]_h\delta_x - f(x)$$

From 2.6

$$_h\Delta_x \cdot [f(x)] \equiv {_h\delta_x} \cdot [f(x)] - f(x)$$

or

$$_h\delta_x \cdot [f(x)] \equiv {_h\Delta_x} \cdot [f(x)] + f(x)$$

Substituting for $_h\delta_x \cdot [f(x)]$ in 2.9 we obtain

$$_h\Delta_x f(x) \overset{S}{=} \{_h\Delta_x \cdot [f(x)] + f(x)\}_h\delta_x - f(x)$$

whence, removing parentheses and rearranging terms,

2.10 $$_h\Delta_x \cdot [f(x)]_h\delta_x \overset{S}{=} {_h\Delta_x} f(x) - f(x)_h\delta_x + f(x)$$

2.10 is a necessary and sufficient condition of 2.6 derived by transformations only of the type described in section 1. Therefore if our product, $f(x)g(x)$, were formally replaced by $Fg(x)$, where F is an operator and $g(x)$ an operand, we could carry through to equation 2.10 without putting any restrictions upon F except that the operators, $_h\Delta_x \cdot F$ and $_h\delta_x \cdot F$, have meaning.

Definition VI:

We define $_h\Delta_x \cdot F$ by the relation

2.11 $$_h\Delta_x \cdot F \overset{S}{=} {_h\Delta_x} F_h\delta_x^{-1} - (F_h\delta_x - F)_h\delta_x^{-1}$$

where $_h\delta_x^{-1}$, the inverse of $_h\delta_x$ has the property

2.12 $$ {}_h\delta_x^{-1} f(x) \equiv f(x - h) $$

Due to the equivalence of 2.6 and 2.11, if F_x is an operator containing the variable, x,

2.13 $$ {}_h\Delta_x \cdot F_x \overset{S}{=} F_{x+h} - F_x $$

and

2.14 $$ {}_h\delta_x \cdot F_x \overset{S}{=} F_{x+h} $$

By 2.5 and 2.11, ${}_h\delta_x \cdot F$ can be expressed by

2.15 $$ {}_h\delta_x \cdot F \overset{S}{=} {}_h\Delta_x F\, {}_h\delta_x^{-1} + F\, {}_h\delta_x^{-1} $$

We must bear in mind that ${}_h\Delta_x P$ means successive applications of F and ${}_h\Delta_x$ whereas ${}_h\Delta_x \cdot F$ means application of the new operator defined by equation 2.11. The same argument applies to ${}_h\delta_x \cdot F$.

3. The Derivative of an Operator:

Consider the classical definition of the derivative:

3.1 $$ Df(x) \equiv \operatorname*{Lim}_{h \to 0} \frac{f(x + h) - f(x)}{h} $$

By 2.4, 2.5, and 2.6 this becomes

$$ Df(x) \equiv L_h \frac{1}{h}\, {}_h\Delta_x f(x) $$

Whence

3.2 $$ D \equiv L_h \frac{1}{h}\, {}_h\Delta_x $$

Definition VII

We define the derivative of an arbitrary operator, F, by the relation

3.3 $$ D \cdot F \overset{S}{=} (L_h \frac{1}{h}\, {}_h\Delta_x) \cdot F $$

provided F is such an operator that the relation has meaning. Then by 2.1 and Definition V

$$ D \cdot F \overset{S}{=} (L_h \frac{1}{h}) \cdot [{}_h\Delta_x F\, {}_h\delta_x^{-1} - (F\, {}_h\delta_x - F)\, {}_h\delta_x^{-1}] $$

$$ \overset{S}{=} L_h \cdot \left\{ \frac{1}{h}\, {}_h\Delta_x F\, {}_h\delta_x^{-1} - \frac{1}{h}(F\, {}_h\delta_x - F)\, {}_h\delta_x^{-1} \right\} $$

3.4 $$ D \cdot F \overset{S}{=} L_h \cdot \left\{ \frac{1}{h}\, {}_h\Delta_x F\, {}_h\delta_x^{-1} \right\} - L_h \cdot \left\{ \frac{1}{h}(F\, {}_h\delta_x - F)\, {}_h\delta_x^{-1} \right\} $$

Definition VIII:

An operator, F, is called continuous if

$$L_h F \overset{S}{=} L_h \cdot F L_h$$

for all continuous operands.

If F and DF are continuous, 3.4 becomes

$$D \cdot F \overset{S}{=} L_h \cdot \left[\frac{1}{h}\ {}_h\Delta_x\right] L_h \cdot (F\ {}_h\delta_x^{-1}) - L_h \cdot \left[\frac{1}{h}(F\ {}_h\delta_x - F)_h\delta_x^{-1}\right]$$

3.6 $$D \cdot F \overset{S}{=} DL_h \cdot (F\ {}_h\delta_x^{-1}) - L_h \cdot \left[\frac{1}{h}(F\ {}_h\delta_x - F)_h\delta_x^{-1}\right]$$

${}_h\delta_x$ is continuous. If F is continuous, and $L_h \cdot \left[\frac{1}{h}(F\ {}_h\delta_x - F)\right]$ also, then

3.7 $$D \cdot F \overset{S}{=} DFL_h\ {}_h\delta_x^{-1} - L_h \cdot \left[\frac{1}{h}(F\ {}_h\delta_x - F)\right] L_h \cdot {}_h\delta_x^{-1}$$

But

$$L_h\ {}_h\delta_x^{-1} \overset{S}{=} {}_D\delta_x^{-1}$$
$$\overset{S}{=} I$$

Thus 3.7 becomes

$$D \cdot F \overset{S}{=} DF - L_h \cdot \left[\frac{1}{h}(F\ {}_h\delta_x - F)\right]$$

The dot following L_h may be dropped since $\frac{1}{h}(F\ {}_h\delta_x - F)$ is assumed continuous. The equation finally becomes

3.8 $$D \cdot F \overset{S}{=} DF - L_h \frac{1}{h}(F\ {}_h\delta_x - F)$$

Definition IX:

A linear operator, A_l, is defined by

3.9 $$A_l(a \pm b) \equiv A_l a \pm A_l b$$

where a and b are permissible operands for A_l, and

3.10 $$A_l n \overset{S}{=} n A_l$$

for all complex numbers, n.

The derivative of a continuous, linear operator is its differential commutator since, by 3.8, 3.9, and 3.10

$$D \cdot A_l \overset{S}{=} DA_l - L_h \frac{1}{h}(A_l \, {}_h\delta_x - A_l)$$

$$\overset{S}{=} DA_l - L_h \frac{1}{h} A_l ({}_h\delta_x - I)$$

$$\overset{S}{=} DA_l - A_l L_h \frac{1}{h} \, {}_h\Delta_x$$

3.11 $$D \cdot A_l \overset{S}{=} DA_l - A_l D$$

Thus we have defined the derivative of an operator so that it fulfills the conditions discussed in section 2. It is easily seen that the derivative of a non-linear functor is the null operator, according to 3.8. The parallelism between operator calculus and ordinary calculus is further borne out by noting that the derivative of a "complex function multiplier" is merely the derivative of that function used as a multiplier also.

4. General Series Representation for Operators:

We define the mean operator, ${}_yM_x$, by the relation

4.1 $${}_yM_x \; f(x) \equiv \frac{f(y) - f(y_0)}{y - y_0}$$

If we attempt successive applications of ${}_xM_x$ we obtain indeterminate forms. So we define the symbol, ${}_xM_x^2$, by the relation

4.2 $${}_xM_x^2 \overset{S}{=} L_{(x \to y)} \; {}_yM_x \; {}_xM_x$$

where $L_{(x \to y)}$ denotes $\lim\limits_{x \to y}$.

Then

$${}_xM_x^2 \, f(x) \equiv L_{(x\to y)} \; {}_yM_x \; {}_xM_x \; f(x)$$

$$\equiv L_{(x\to y)} \; {}_yM_x \left[\frac{f(x) - f(x_0)}{x - x_0}\right]$$

$$\equiv L_{(x\to y)} \frac{1}{y - y_0} \left[\frac{f(y) - f(x_0)}{y - x_0} - \frac{f(y_0) - f(x_0)}{y_0 - x_0}\right]$$

$$\equiv \frac{1}{x - x_0}\left[\frac{f(x) - f(x_0)}{x - x_0} - {}_{x_0}Df(x)\right]$$

$${}_xM_x^2 \, f(x) \equiv \frac{1}{x - x_0} \left[{}_xM_x \; f(x) - {}_{x_0}Df(x)\right]$$

or

4.3 $$_xM_x^2 \overset{S}{\equiv} \frac{1}{x - x_0}\ [_xM_x - {}_{x_0}D]$$

where $_{x_0}D^n$ has the meaning of the nth derivative with x replaced by x_0.

Similarly we define $_xM_x^3$ by

4.4 $$_xM_x^3 \overset{S}{\equiv} L_{(x\cdot y)}\ {}_yM_x\ {}_xM_x^2$$

i.e.

$$\begin{aligned}
_xM_x^3\ f(x) &\equiv L_{(x\cdot y)}\ {}_yM_x\ {}_xM_x^2\ f(x) \\
&\equiv L_{(x\cdot y)}\ {}_yM_x\ \left[\frac{1}{x - x_0}\left\{\frac{f(x) - f(x_0)}{x - x_0}\right\} - \frac{1}{x - x_0}\ {}_{x_0}Df(x)\right] \\
&\equiv L_{(x\cdot y)}\ \frac{1}{y - y_0}\left[\frac{f(y) - f(x_0)}{(y - x_0)^2} - \frac{_{x_0}Df(x)}{y - x_0} - \frac{f(y_0) - f(x_0)}{(y_0 - x_0)^2} + \frac{_{x_0}Df(x)}{y_0 - x_0}\right] \\
&\equiv \frac{1}{x - x_0}\left[\frac{f(x) - f(x_0)}{(x - x_0)^2} - \frac{_{x_0}Df(x)}{x - x_0} - \tfrac{1}{2}\ {}_{x_0}D^2f(x)\right] \\
&\equiv \frac{1}{x - x_0}\ ({}_xM_x^2 - \tfrac{1}{2}\ {}_{x_0}D^2)f(x)
\end{aligned}$$

Iteration of this process gives

4.5 $$_xM_x^{n+1} \overset{S}{\equiv} \frac{1}{x - x_0}({}_xM_x^{n-1} - \frac{1}{n!}\,{}_{x_0}D^n)$$

Substituting the corresponding equation for $_xM_x^n$ into 4.5,

$$_xM_x^{n+1} \overset{S}{\equiv} \frac{1}{x - x_0}\left[\frac{1}{x - x_0}({}_xM_x^{n-1} - \frac{1}{(n-1)!}\ {}_{x_0}D^{n-1}) - \frac{1}{n!}\ {}_{x_0}D^n\right]$$

Substituting in the above equation with the corresponding equation for $_xM_x^{n-2}$ and continuing this process,

$$_xM_x^{n+1} \overset{S}{\equiv} \frac{1}{(x - x_0)^n}\ {}_xM_x - \sum_{i=1}^{n} \frac{_{x_0}D^i}{i!\,(x - x_0)^{n-i+1}}$$

4.6 $$(x - x_0)^{n+1}\,{}_xM_x^{n+1} \overset{S}{=} (x - x_0)\,{}_xM_x - \sum_{i=1}^{n} \frac{(x - x_0)^i\,{}_{x_0}D^i}{i!}$$

But

$${}_xM_x f(x) \equiv \frac{f(x) - f(x_0)}{x - x_0}$$

$$\equiv \frac{1}{x - x_0}[I - {}_{x_0}D^0]\,f(x)$$

or

4.7 $${}_xM_x \overset{S}{=} \frac{1}{x - x_0}[I - {}_{x_0}D^0]$$

Whence, substituting 4.7 in 4.6 and rearranging terms

4.8 $$I \overset{S}{=} \sum_{i=0}^{n} \frac{(x - x_0)^i\,{}_{x_0}D^i}{i!} + (x - x_0)^{n+1}\,{}_xM_x^{n+1}$$

We see that this is an operator derivation of the Taylor expansion. The operator on the right hand side of 4.8 we shall call the Taylor operator of order, n, denoted by T_n.

Using the symbolic operation of both sides of 4.8 on an operator possessing higher derivatives,

4.9 $$A \overset{S}{=} \sum_{i=0}^{n} \frac{(x - x_0)^i\,{}_{x_0}D^i \cdot A}{i!} + (x - x_0)^{n+1}\,{}_xM_x^{n+1} \cdot A$$

By a process similar to the development of equation 4.5 it can be shown that the remainder, $(x - x_0)^{n+1}\,{}_xM_x^{n+1} \cdot A$, can be put in the standard Newtonian form involving the $(n + 1)$st derivative of the operator, A, at some point, x_r, between x and x_0. If

4.10 $$L_{\frac{1}{n}} \cdot [(x - x_0)^{n+1}\,{}_xM_x^{n+1} \cdot A] \overset{S}{=} \Omega$$

Then the operator, A, is expressible by the series

4.11 $$A \overset{S}{=} \sum_{i=0}^{\infty} \frac{(x - x_0)^i\,{}_{x_0}D^i}{i!} \quad .$$

ARITHMETIC - ALGEBRA - GEOMETRY EACH AS AN AID TO THE STUDY OF THE OTHER

W. W. Rankin

We are in session as a group of teachers of mathematics.* This being interpreted properly would indicate that we are concerned with two aspects of mathematics - greater clarity as to the structure of mathematics and its significance in our current living - better artistry in classroom performance. At the level of our interest we may think of mathematics as dealing with three important ideas: number, quantity, and space (form).

Experience will warrant the postulate that all persons of normal intelligence have some interest in: number, quantity, space. I should hasten to add though that many soon lose their interest in the formalism required in the traditional study of mathematics.

With nature continuously dramatizing through her periodicities the process of counting, it is difficult to understand how it required so long for man to develop a number system. The Hindu-Arabic numerals along with the place value concept will always rank as a major contribution to civilization. Europe was struggling along trying to study algebra with Roman numerals until the 14th century.

For the purpose of discussion it might be good at this point to build some very thin partitions between: mathematician - scientist - industrialist (including business men and engineers). A mathematician by virtue of his interests and his methods of work is much concerned with the possible orders in which things (or ideas) fit together. Because of his trustworthy methods and his fidelity of purpose he enjoys a place of esteem wherever precision of thought is of importance. By the manner in which he arrives at his conclusions and the meticulous care he exercises in arriving at his conclusions he is able to make predictions. This makes great savings of time and expense to the scientist and to the industrialist. To be sure this adds to his stature and pride.

A scientist is committed to finding the *actual order* in which things fit together. He is heir to the achievements of the mathematician and has added his own developments in measuring. He speaks casually of a millionth of an inch, a millionth of a second, and offers to industry refinements of measuring which in turn industry passes on to the consumer in better and more precise instruments and machines. Scientists have just cause to be proud of the many secrets they have coaxed from mother nature. The actual order in which things fit together gives to the

*A talk given to the Mathematics Institute at U.C.L.A., July 1951.

scientist many opportunities to exhibit to laymen his accomplishments.

Industrialists (inclusing the business men and engineers) are necessarily given to finding the *useful order* of fitting things together. They must fit things together in such a manner that they may sell their product at a profit. In turn they are properly called upon to share part of this profit as a subsidy to their benefactors - mathematicians and scientists - for their contributions to the "know how" of industry. Industrialists are in constant contact with the work-a-day world. They literally "make the wheels go round". It is they who take our formulas and turn them back into arithmetic before producing dividends, engines, television sets, etc. It is they who provide our necessities, our comforts and our luxuries. If this group chooses to speak of mathematics as a "tool" we must accept this in the spirit and understanding in which this is said, for to them it as a "tool" .

The above thin partitions were constructed to get teachers to realize that in a class of 30 students, of those who are interested in mathematics, the interest may have varying shades of these three aspects of mathematics.

It is claimed by some that the wheel concept has been man's greatest emancipator from his inherent physical limitations. In human and in animal life the essential and natural motion is hinge motion. But slowly man discovered he could project himself out of his endowed and restricted limitation of hinge motion into circular motion. Here we get a glimpse of man in his early efforts to abstract. A little reflection on the wheel's place in our present technological society will give some interest to this idea.

A new era was ushered into mathematics when Descartes (1637) abstracted from the wheel (circle) $X^2 + Y^2 = R^2$. It is precisely this abstract quality of mathematics that helps to characterize mathematics and we grow to feel the pull of its great power to set forth relationships between ideas and to set them forth without emotional or economic disturbances. These laws of nature, $s = \frac{1}{2}gt^2$, $x^2/a^2 + y^2/b^2 = 1$, $y = e^{kx}$, are but exhibits of man's abstractions. Countless other exhibits of abstractions might be shown. As a matter of psychological considerations I feel that more of our abstractions used in the class room should come from interesting relationships of nature and current life situations. Our textbooks need to provide more illustrations of actual and useful relationships in the lists of problems.

The critical thinking which is possible within postulational systems has given to man a very just reason to admire his own achievements. It also gives him a type of mental security where he can determine the restrictions placed upon his knowledge. He can say "I know this is true within the limitation of the postulates I have assumed." Some graduate work in "Foundations of Mathematics" would serve to strengthen the background for teachers in secondary mathematics. On the high school level and early college work we shall not have opportunity to exhibit much of non-Euclidean geometry or Peano's postulates for arithmetic, etc. In the teacher's mind though they do serve as good examples of critical thinking.

The teacher, if an artist in the classroom must constantly have an awareness of the possible - the actual - the useful orders of fitting things together. There is always the temptation to be content with the useful order. Students are coming up through the high school and on through college mathematics with very little appreciation of what the significance is of the one-to-one correspondence in mathematics and hence fails to observe the relationships between arithmetic - algebra - geometry - a lack of power results.

The *function concept* has been very well described as the "declaration of dependence". Things, ideas, and human affairs seem to be so inextricably interrelated that we may very properly feel that the function concept is the most important idea in the development of mathematics, from both the abstract and the concrete points of view. We are inclined to treat this as a special topic rather than an all pervasive idea running throughout mathematics. Our techniques for treating this important concept need to be revised or better still completely reorganized. It is through the function concept that we are able to abstract and express in simple form so many relationships of: number, quantity, form. Many of these relationships deal with problems of the daily affairs of life. With a moderate amount of understanding of the language of mathematics - symbols, graphs, etc. a person can gain a much clearer idea of how precisely these relationships do fit together. Under our present method of treating this whole idea students do not gain much taste for or skill in using to full advantage the function concept.

Our traditional offering of the function concept to students is through the following four methods: (a) verbal statement, (b) equation, (c) graph, (d) table. Perhaps 80% of the emphasis is on the (b) equation and the manipulative processes involved in the study of equations. The story told by a quadratic function is much more instructive than the quadratic equation. It may include the quadratic equation. This is especially true if the quadratic function is also studied graphically. If we ask a student to investigate a quadratic or cubic function - on his own resources he has at least a chance to cultivate the "spirit of discovery". It is interesting to speculate - what he might learn as compared with what he actually does learn in solving numerous quadratic equations. In geometry the function concept is not consciously developed and hence little power in mathematical analysis is gained in the traditional study of geometry. Students grow up to feel that geometric methods are totally different from the methods of arithmetic and algebra. A common comment from students after completing a course in geometry is "I finished geometry in the 10th grade" (a severe criticism of the teacher).

It seems to me that (c) the graph offers the best opportunity to present the function concept as well as many other aspects of mathematics, and to bring about both a unity and a comprehension of mathematics which we are unable to get by the partitioning of mathematics into arithmetic, algebra, geometry, trigonometry, etc. The use of the hand and the use of the eye offer psychological advantages which we need to recognize and evaluate. Students can be trained to abstract many relationships from the graph once he has an insight into the meaning of a one-to-one correspondence - between arithmetic, algebra, geometry. The graph tends to focus with clarity the concepts of number, quantity and form. Indeed the graph does furnish a "royal road to geometry". Some of our most difficult concepts such as: ratio, congruence, continuity, limit, periodicity find excellent expression in the use of the graph. One of the real bottlenecks in the study of mathematics lies in the study of three dimensions. Training in drawing three dimensional figures and in constructing three dimensional models would serve to make the proposed ideas much clearer, and in turn aid in making the abstractions desired. Ask a student to graph $E = \frac{1}{2}Mv^2$, E = destructive force of a car M = mass = W/G, v = velocity. Then compare the value of E for speeds of 20, 40, 60, 80 m.p.h. This abstraction could aid greatly in promoting safe driving.

The complex numbers rapidly gained acceptance after Wessel (early 19th century) showed that they could be represented graphically. More recently we observe they may be advantageously represented on wire models. A student feels much more friendly (a necessary prerequisite for satisfactory learning) towards complex numbers when he discovers he can "put his finger" on a complex number. The uses of complex numbers in physics and electronics and elsewhere when magnitude and direction are significant, certainly points the way for more study of complex numbers in elementary mathematics. The completeness it offers to many ideas of algebra renders greater powers of abstraction.

Our daily doings are so completely arithmetical that the transition into the general from the particular is more difficult than commonly thought. Please hear A. N. Whitehead "To see what is general in what is particular, and what is permanent in what is transitory is the aim of modern science". The use of some adjustable instrument (geometry) will aid greatly in establishing the "any" concept. We need some very simple aids with which we can bring the student to the "moment of insight". The model or instrument lingers to hold the attention long after the spoken word has slipped away. To some this "moment of insight" comes quickly and vividly, to some it comes slowly and often very vaguely. Clarity of understanding is closely related to precision of statement, and for this reason I feel there should be frequent calls for verbal statements. Because of the tempo of current living we must find more artistic ways of presenting ideas, and by this I mean to include ways and means for more rapid comprehension on the part of students. At a recent meeting of our Committee on Coordination of Mathematics with Business and Industry we were told by consultants from industry "you must move out beyond black and white and the spoken word, if you are to get and hold the attention of your customer".

May I offer you please the following striking story in symbolic form setting forth some of the development of the *ratio concept*:

$$20/60 = 1/3 = .3333... = a/b = \tan \theta$$

$$= (y_2 - y_1)/(x_2 - x_1) = dy/dx$$

Here we may view the ratio concept as it evolved through 3000 years. The 60 in the denominator of the first fraction is a Babylonian contribution. They used fractions having only 60 or some multiple of 60 for the denominators. The second fraction with its 1 in the numerator is an Egyptian idea. They used only fractions having 1 in the numerator. The decimal fraction idea was largely that of Stevin (1685). The remainder of the above story is too well known to relate here. In this exhibit we find a close relationship between arithmetic, algebra, trigonometry, analytical geometry, and calculus. It is as a silver thread running through these subjects. A careful study of the many ways devised to avoid fractions would be very beneficial to any teacher of mathematics. It would build up a proper type of sumpathy for the students who struggle with the ratio concept. As a part of this story it should be noted that the dy/dx rode into mathematics on the back of geometry as it was applied to the study of motion. By this I mean on the back of analytical geometry. There is a tendency to pass lightly over this dynamic aspect of mathematics. With this new dy/dx concept most of the earlier mathematics was hastily verified and then the mathematicians turned to find new worlds to conquer. And verily there was a conquest, for more mathematics was learned in the second half of the 17th century than had been known in the preceding 3000 years. Not only had mathematics found a new way of thinking, but this readily became a powerful "tool" for the scientists and the industrialists.

It will require skill, patience, and artistry on the part of the teacher to find which of the many "Teaching aids" are worthwhile and how to make proper use of them. At present I should like to advise these teaching aids be painted both green and red in order to give the "go" and the "stop" signals as to when to use them. It takes real skill to abstract from these aids the mathematical principles associated with the aids, and until this is done the teacher's job lacks in cleverness and real worth.

From experience I would like to offer two very simple and quite different illustrations of relating number, quantity, and form.

1. Use a plyboard wheel (about 15 inches in diameter) with a notch in the rim to hold a piece of chalk. Roll the wheel along the chalkrail against the blackboard. The chalk in the rim of the wheel of course

will describe a cycloid. The parametric equations ($x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$) can easily be developed after a little trigonometry has been acquired. For the student in analytical geometry it is possible and highly desirable that he obtain a friendly feeling for the idea of parameter which he could do here. A more vivid presentation can be done in 10 minutes than can be done in 20 minutes without this aid. It is extremely difficult for students to study motion from hastily drawn figures on the blackboard. It is easy to repeat the motion with an instrument if this seems desirable.

2. As a second illustration showing the interplay of arithmetic, algebra, and geometry I will suggest this simple problem. It is known that 3, 4, 5 will form the sides of a right angle triangle. Are there other sets of consecutive integers? Algebra suggests that we set down such a set as these: $n - 1$, n, $n + 1$, applying the Pythagorean theorem we have

$$(n + 1)^2 = (n - 1)^2 + n^2 \text{ or } n(n - 4) = 0$$

$n = 4$, $n = 0$ (trivial). Thus leading us back to 3, 4, 5. In this clever way algebra announces with assurance there is no other set of consecutive numbers which will form a right angle triangle.

One other suggestion, please set, $x^2 + y^2 = r^2$; now multiply both sides by $\pi/4$ giving $(\pi/4)x^2 + (\pi/4)y^2 = (\pi/4)r^2$ but with this simple operation we show that the area of the circle on the hypotenuse is equal to the sum of the areas of the other circles on the other two sides of the triangle as diameters.

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INFINITE SERIES AND TAYLOR AND FOURIER EXPANSIONS

Robert C. James

1. *Sequences and series of constant terms.* One of the most fundamental and important problems of mathematics is that of approximating some quantity which can not be exactly evaluated or exactly expressed in the desired form. Examples of this are problems of finding an approximate decimal representation for an irrational root of an equation, evaluating trigonometric functions and logarithms, and evaluating definite integrals or solving differential and integral equations which are not readily integrable. An approximation has value only if one knows how close an approximation it is, while a method of approximation is of value only if it will yield as accurate an approximation as one may desire. As an example, consider the problem of expressing $\sqrt{2}$ as a decimal. Since $(1.4)^2 < 2$ and $(1.5)^2 > 2$, one can conclude that $\sqrt{2} = 1.4$ with an error of less than .1. Likewise, $\sqrt{2} = 1.41$ with an error of less than .01. This process yields a sequence of numbers $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, ... which approaches $\sqrt{2}$ in the sense that if one specifies how close an approximation to $\sqrt{2}$ is desired, then each term beyond a certain one will be at least this close to $\sqrt{2}$. For example, if one wishes to evaluate $\sqrt{2}$ with an error of less than .00001, then $a_5 = 1.41421$, or any term after a_5, will give such an evaluation. The number π has been approximated by various means since the problem of computing the circumference of a circle was first studied. Some early approximations were very good, others very poor, while most of them were given with no determination of their true accuracy. Since π is defined as the ratio of the circumference of a circle to its diameter, the circle C_1 with unit diameter has a circumference of length π. One of the most elementary methods of approximating π is to approximate the circumference of this circle by the perimeter of a regular inscribed polygon. This can be done by use of the formula $s_{2n} = [\frac{1}{2}(1 - \sqrt{1 - s_n^2})]^{\frac{1}{2}}$, which gives the length s_{2n} of a side of a regular polygon of $2n$ sides inscribed in C_1 in terms of the length s_n of a side of a regular polygon of n sides. Since a regular hexagon inscribed in C_1 has a perimeter $p_1 = 3$ with $s_6 = \frac{1}{2}$, a regular polygon of 12 sides has a perimeter $p_2 = 12[\frac{1}{2}(1 - \sqrt{1 - \frac{1}{4}})]^{\frac{1}{2}} = 3.106 \cdots$. Continuing this gives a sequence $p_1, p_2, p_3, \cdots$ of increasing numbers, each less than π. However, it is not easy to show how close an approximation of π is given by a certain term of the sequence. One way of doing this would be to find the perimeter of circumscribed polygons in a similar way, which would give a sequence $q_1, q_2, q_3, \cdots$ of

decreasing numbers, each greater than π. If one wants to find a p_n which differs from π by less than some allowable error ϵ, it would then be sufficient to find a p_n differing from the corresponding q_n by less than ϵ. By use of calculus, one can derive other methods of approximating π. For example, the sequence $s_1 = 1$, $s_2 = 1 - 1/3$, $s_3 = 1 - 1/3 + 1/5$, $\cdots$ approaches $\pi/4$. In fact, the error in approximating $\pi/4$ by s_n is less than $1/(2N + 1)$ if $n \geq N$. This can also be expressed symbolically by: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$. This indicated sum of an infinite number of terms is called an *infinite series*. Another series representation of π is given by:

$$\frac{\pi}{4} = \left[\frac{7854}{10,000} - \frac{1}{545,261}\right] - \frac{1}{3}\left[\frac{7854}{(10,000)^3} - \frac{1}{(545,261)^3}\right]$$
$$+ \frac{1}{5}\left[\frac{7854}{(10,000)^5} - \frac{1}{(545,261)^5}\right] - \cdots.$$

This means that for any allowable error ϵ there is a number N such that if one adds up at least N terms of this series the sum will differ from $\pi/4$ by less than ϵ. For a given ϵ, N can be much smaller for the second series than for the first.

A precise definition of what is meant by the limit of a sequence or by the sum of a series is necessary if one is to develop a mathematical theory involving these concepts. The following definitions serve this purpose, though they merely state in concise mathematical terms the intuitive meaning of limit and sum discussed above.

A sequence $a_1, a_2, \cdots$ *is said to be convergent if there is a number L, called the limit, such that for any* $\epsilon > 0$ *there is a number N for which* $|L - a_n| < \epsilon$ *if* $n \geq N$. *This is expressed symbolically by* $\lim_{n\to\infty} a_n = L$.

A series $a_1 + a_2 + a_3 + \cdots$ *is said to be convergent if there is a number S, called the sum, such that for any* $\epsilon > 0$ *there is a number N for which* $|S - s_n| < \epsilon$ *if* $n \geq N$, *where* s_n *is the sum of the first n terms of the series. This is expressed symbolically by* $\sum_{n=1}^{\infty} a_n = S$.

The concepts of convergence of a sequence and convergence of a series are closely related, the convergence of a series $a_1 + a_2 + a_3 + \cdots$ being equivalent to the convergence of the sequence of partial sums $s_1, s_2, \cdots$, while the convergence of a sequence $s_1, s_2, \cdots$ is equivalent to the convergence of the series $a_1 + a_2 + \cdots$, where $a_n = s_n - s_{n-1}$. Thus each theorem about convergence of a sequence corresponds to an analogous theorem about series, and conversely. For simplicity, only the language of series will be used hereafter.

It is seldom practical to sum the terms of an infinite series to prove convergence of the series. Methods such as those given by the theorems discussed below are usually necessary. A striking exception is the geometric series $a + ar + ar^2 + \cdots$, for which $s_n = a + ar + \cdots + ar^{n-1} = a(1 - r^r)/(1 - r)$. If $|r| < 1$, then $\lim_{n\to\infty} s_n = a/(1 - r)$. This means that the series converges and has the sum $a/(1 - r)$.

A series $a_1 + a_2 + \cdots$ of non-negative terms is convergent if the sequence of partial sums $s_1, s_2, \cdots$ is bounded, that is, if there is an *upper bound* M such that $s_n \leq M$ for each n. The proof of this theorem is immediate if it can be shown that there is a least number $\overline{M}$ such that $s_n \leq \overline{M}$ for each n. For if $\overline{M}$ is the least such number, then the terms of the sequence must eventually get arbitrarily close to $\overline{M}$. In other words, for any $\epsilon > 0$ there is an N such that $\overline{M} - \epsilon < S_N$. Since the terms of the sequence are non-negative, s_n cannot decrease as n increases. Hence $|\overline{M} - s_n| < \epsilon$ for each $n \geq N$. Thus the proof of this theorem is dependent on the fact that a set of numbers which has an upper bound has a least upper bound. The latter can be rigorously proven only with use of a careful definition of irrational numbers. It is one of many ways of characterizing the *completeness* of the real number system – intuitively, that for any series which behaves like a convergent series there exists a real number which is the sum. This is expressed in another way by the following:

Cauchy's Theorem. A necessary and sufficient condition for convergence of a series $a_1 + a_2 + \cdots$ is that for any $\epsilon > 0$ there is an N such that $|a_n + a_{n+1} + \cdots + a_{n+p}| < \epsilon$ if $n \geq N$ and $p \geq 0$.

The necessity of the condition of this theorem is a direct consequence of the definition of convergence. The sufficiency can be established by showing that the sequence of partial sums $s_n, s_{n+1}, \cdots$ has a greatest lower bound $\underline{S}_n$ and that the increasing sequence $\underline{S}_1, \underline{S}_2, \cdots$ has a least upper bound S, which can be shown to be the sum of the series. Many other proofs of this fundamental and important theorem can be given. The condition of the theorem is sometimes used as the definition of convergence, the existence of a sum for the series then being proven.

A series $a_1 + a_2 + \cdots$ is convergent if there is a convergent series of non-negative numbers $r_1 + r_2 + \cdots$ such that $|a_n| \leq r_n$ for each n. This test for convergence is called the *comparison test*. It is an immediate consequence of Cauchy's theorem, since

$$|a_n + a_{n+1} + \cdots + a_{n+p}| \leq |a_n| + |a_{n+1}| + \cdots + |a_{n+p}| \leq r_n + \cdots + r_{n+p}$$

implies that the series $a_1 + a_2 + \cdots$ satisfies the condition of Cauchy's theorem by virtue of this condition being satisfied by the series $r_1 + r_2 + \cdots$. A series $a_1 + a_2 + \cdots$ is said to be *absolutely convergent* if the series $|a_1| + |a_2| + \cdots$ is convergent. It is clear from the comparison test that *an absolutely convergent series is* *convergent.*

If, for some number N, $|a_{n+1}/a_n| \leq r < 1$ for some fixed number $r < 1$ and each $n \geq N$, then the series $|a_1| + \cdots + |a_N| + r|a_N| + r^2|a_N| + r^3|a_N| + \cdots$ will serve as the comparison series $r_1 + r_2 + \cdots$ of the above comparison test. Thus the series $a_1 + a_2 + \cdots$ is convergent. If, for some number N, $|a_{n+1}/a_n| \geq 1$ for each $n \geq N$, then $\lim_{n\to\infty} a_n \neq 0$ and the series is divergent. These criteria for convergence in terms of the behavior of the ratio $|a_{n+1}/a_n|$ constitute the *ratio test.* A more sensitive test of this type can be obtained by analysing the case in which $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. This can be done by comparison with the series $1 + 1/2^p + 1/3^p + \cdots$ for which $\lim_{n\to\infty} a_{n+1}/a_n = 1$ and which converges if $p > 1$ and diverges if $p \leq 1$. This leads to the result that a series $a_1 + a_2 + \cdots$ is convergent if there is a number $p > 1$ and a number N such that $|a_{n+1}/a_n| \leq 1 - p/n$ for each $n \geq N$; the series is divergent if there is a number $p \leq 1$ and a number N such that $|a_{n+1}/a_n| \geq 1 - p/n + f(n)/n^2$ for each $n \geq N$, where $f(n)$ is bounded. Many other tests for convergence of infinite series could be given.

2. *Series of variable terms.* All of the series of the above discussion were series each of whose terms were constants. If x is given a particular value in each term of a series $u_1(x) + u_2(x) + \cdots$ whose terms are functions of a variable x, the series becomes a series of constants and the concept of convergence already discussed is applicable. Such a series may converge for certain values of x and diverge for other values of x. The sum of the series will be a function $S(x)$ of x, whose domain of definition consists of all x for which the series has a sum. A series of constants can be used to compute particular quantities, such as π or $\sqrt{2}$. A series of variable terms represents a function in a form that is frequently very useful, for example in such problems as evaluating integrals and solving differential and integral equations.

(To be concluded in the next issue.)

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Introduction to the Theory of Statistics. By Alexander McFarlane Mood, McGraw-Hill Book Company, Inc., New York, 1950, XIII + 433 pages, $5.00.

Within the last few years a fairly large number of books have appeared on special phases of statistics, but textbooks for college classes with a calculus background and commencing the study of mathematical statistics have been comparatively few. One of the best of these latter is the text under review. It should, therefore, find wide acceptance by departments of mathematics wishing to offer a strong course in this subject. Mathematical topics beyond elementary calculus are developed as needed, for example, a brief discussion of the theory of sets and something of the algebra of matrices. No knowledge of probability being assumed, the book commences with this subject, which is followed by the development of mathematical models that approximate experimental situations. Statistical inference and design of experiments are treated last.

As its title implies, the book's emphasis is upon theory. There is no discussion of descriptive statistics and few calculation problems based on tables of numerical data. Stimulating problems occur at the close of each chapter except the first one. While there are more than five hundred of these, a few more solved, illustrative problems in the text might have been helpful to the weaker student. The problems often develop further the theory of the text, for example, correlation is treated almost entirely in the problems. No answers to problems are given in the text, but a separate answer pamphlet with all answers is available. No wrong answers were found in a sample of problems checked. The tables assembled at the back of the book for problem solving and for reference, while not numerous, are adequate. They consist of tables for the normal, chi-square, Student's t and F distribution functions. More references to original sources, given throughout the text, would doubtless have been welcome to some readers. There are no footnotes, but the references given are accompanied by very pertinent comments.

Some other characteristics of the book, chosen rather at random, are: new ideas are introduced with concrete examples; emphasis is laid on marginal and conditional distributions; expected values are used to introduce moments and moment generating functions; a proof is given for a restricted form of the theorem that identical moment generating functions imply identical densities; deduction of the normal approximation to the binomial distribution is excellent; the normal distribution is treated for n variables; the principle of maximum likelihood is explained with special clarity; independence of the sample mean and sample variance is not assumed but rigorously established for normal populations; a thorough exposition of the Neyman-Pearson theory of testing hypotheses is given as well as a similar treatment of the analysis of variance.

Each page is numbered and also marked with chapter number and section number. This should prove a convenience. There are seventy-four well-drawn figures. Some misprints were noted, but they were easily recognized as such. Doctor Mood's book is a careful development of the basic ideas of modern statistical theory.

University of Arizona R. F. Graesser

Brief Course in Analytics. Revised Edition. By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1951. XI + 224 pages. $2.40.

This text is a revision of an earlier edition of 1940. The authors have replaced some of the problems of the earlier edition and increased the number. Each set of problems progresses from the very easy to those which are more thought provoking, thus affording a challenge to the better student.

In this edition an introductory chapter of basic formulas, tables of logarithms, and natural trigonometric functions have been added. The text is small in terms of physical dimensions and is designed for a three semester hour course.

The content of the text does not differ from the traditional one on this subject. A claim to uniqueness might be made in the arrangement of topics in that some general second degree equations, extent of a curve, and translation of axes are studied before taking in detail the straight line and conic sections. The reviewer considers this arrangement a debatable pedagogical practice.

McNeese State College W. H. Bradford

MISCELLANEOUS NOTES

Edited *by*

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

A GEOMETRIC PERPETUAL CALENDAR

In 1952 we have the two hundredth anniversary of the adoption of the Gregorian Calendar by England and her colonies. Perhaps this justifies the introduction of a new perpetual calendar, a scheme by means of which the day of the week corresponding to a given date can be determined. As further justification, we submit that the scheme below is easier to remember than other perpetual calendars. Those who disagree will no doubt agree, at least, that such schemes emphasize the need for calendar reform similar to that achieved by the "World Calendar"[1], a calendar endorsed by the Mathematical Association of America.

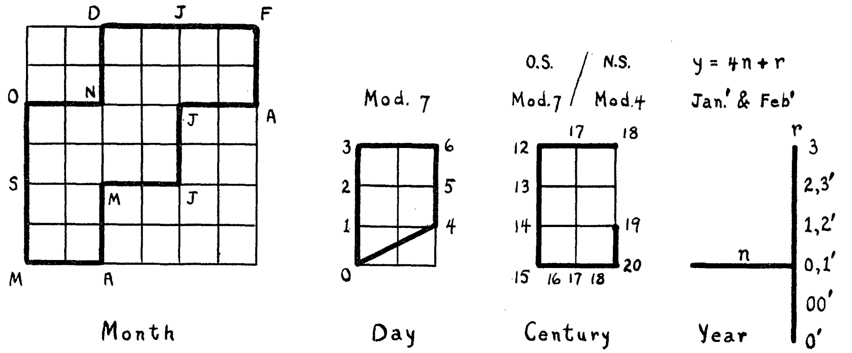
First, a dash of history. The Julian Calendar, first used in 45 B.C., initiated a regular leap year. This calendar fitted the seasonal year very well for a long time, but by the sixteenth century the two were ten days out of phase. Pope Gregory XIII and his astronomers proposed that October 4, 1582 be followed by October 15, 1582 and that from 1582 on there be only 97 leap years every 400 years - the century years not divisible by 400 being dropped as leap years. The Gregorian Calendar was not used by the American Colonies until 1752. At that time there was a difference of 11 days between the "Old Style" and the "New Style" calendars.

We now describe a method for determining the day of the week corresponding to a given date (say, October 16, 1582). The date splits into four parts: the month (October), the day (16), the century (15), and the year (82). We graph four letters as shown in the accompanying figure: *M* for month, *D* for day, *C* for century, and *Y* for year. We note century 15 refers to the 1500's, not to the 15th century.

Consider *M*. March (the first month beginning with *M*) is placed at the origin. The months from April to August zigzag in order from (2,0) to (6,4). The months from September to January zigzag in similar fashion over the northern edge of *M* from (0,2) to (4,6). February at (6,6) completes the *M*. (Note the positions of March and September. Equinoxiously speaking, we may spring into fall or fall into spring). The unduly high ordinates thus given January and February are compensated for by lower ordinates given these months in the letter *Y*.

[1]For details of this calendar and arguments in favor of its adoption, write to "The World Calendar Association, Inc., 630 Fifth Avenue, New York 20, New York."

Consider D. Days of the month are taken modulo 7. Day 0 is placed at the origin. Days 1 to 3 have positions from (0,1) to (0,3), days from 4 to 6 have positions from (2,1) to (2,3).



Consider C. Centuries in the Julian Calendar are taken modulo 7. Century 15 (the beginning of the end of this calendar) is placed at the origin. Centuries 12 to 18 (Old Style) run counterclockwise from (0,3) to (3/2,0), dropping by units to century 15 and then moving at half speed to the right. Thus, with C as a clock face, century 17 (Old Style) is at six o'clock. Century 17 (New Style) is at twelve o'clock. (Remember this is the century the colonies moved up from the old to the new calendar). Centuries in the Gregorian Calendar are taken modulo 4. They progress clockwise around C from century 17 to century 20 at the point (2,0). There is, of course, a double space between the last and the present century in this sequence.

Consider Y. The year y is taken in the form $4n + r$, where r is 0, 1, 2, or 3. It is placed at the point (n, r) except when the date involves January or February. For these months, it is placed at the point (n, r') where, for $r \neq 0$, $r' = r - 1$. In general, $0'$ is -2. In the exceptional case, denoted by $00'$, its value is -1; this is used for dates in the Gregorian Calendar where the century-year number is divisible by 100 but not by 400. (In other words, for the years which have $r = 0$ but which are not leap years)[2].

The actual computation is simple. Each of the four parts of a date now corresponds to a point. Each point has as coordinates the components of the vector going from the origin to that point. Add these four vectors. Dot the sum with the vector $(-2,1)$; that is, multiply the first component by -2, the second component by 1, and add. The resulting number represents the day of the week modulo 7 where the first day of the week, Sunday, corresponds to the number 1, and so on.

Those to whom the validity of the process is not self-evident may verify it by mathematical induction. To remember the process requires recalling the vector $(-2,1)$. It helps to remember that $(2,1)$ is the

[2]The special treatment given January and February can be avoided by regarding these as terminal months of the previous year.

vector corresponding to the present century. Another method is the following. The first vector in the product is remembered as "month, day, century, year, add" , or "*MDCYA*" , or $f(Y)$. The second vector is $f(P)$, or "*MDCPA*" , or "minus deuce comma plus ace" .

We give two examples:

A. December 7, 1941 was a Sunday since

$$(2,6) + (0,0) + (2,1) + (10,1) = (14,8) \text{ and}$$

$$(14,8) \cdot (-2,1) = -28 + 8 \equiv 1 \pmod{7}.$$

B. January 18, 1800 was a Saturday since

$$(4,6) + (2,1) + (2,3) + (0,-1) = (8,9) \text{ and}$$

$$(8,9) \cdot (-2,1) = -16 + 9 \equiv 0 \pmod{7}.$$

This is the same day as January 7, 1800 (Old Style). As a check, $(4,6) + (0,0) + (3/2,0) + (0,-2) = (11/2,4)$ and $(11/2,4) \cdot (-2,1) \equiv 0 \pmod{7}$.

Washington University Marlow Sholander

OMEGA

Cube root of one. Without ever solving the equation $x^3 = 1$ in the usual way, we can sport about with the cube roots of 1 in an interesting manner. Since $1 \times 1 \times 1 = 1$, it is certain that 1 is a cube root of itself, but may there not be some other? If 1 has another cube root, distinct from 1, let us call it ω, omega. Since $(\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1^2 = 1$, it follows that ω^2 is also a cube root of 1.

Other cube roots. If we assume that there is another cube root, v, (distinct from 1, ω, or ω^2), $v^3 = 1$, and we must have v satisfy the equation $v^3 - 1 = 0$. If v is distinct from 1, then $v - 1 \neq 0$ and we can divide the equation by $(v - 1)$ and obtain the new equation $v^2 + v + 1 = 0$. Again if v is distinct from ω, $v - \omega \neq 0$, and we can divide this new equation by $v - \omega$, as follows:

$$\begin{array}{rl}
\underline{v - \omega)} & v^2 + v \qquad\quad + 1 \;\; \underline{(v + (1 + \omega)} \\
 & \underline{v^2 - v\omega} \\
 & \qquad v(1 + \omega) + 1 \\
 & \qquad \underline{v(1 + \omega) - \omega(1 + \omega)} \\
 & \qquad\qquad\qquad 1 + \omega + \omega^2
\end{array}$$

The remainder is the sum of the three roots already considered, and the question arises whether this remainder is zero or not.

The sum of the first three roots. If we put $S = 1 + \omega + \omega^2$, and add the three equations

$$S = 1 + \omega + \omega^2$$
$$\omega S = \omega + \omega^2 + 1$$
$$\omega^2 S = \omega^2 + 1 + \omega$$

we get $S^2 = 3S$, whence either $S = 0$, or else we can divide by S and get $S = 3$. But if $S = 3$, $3 = 1 + \omega + \omega^2$ and $3\omega = \omega + \omega^2 + 1$, that is $3\omega = S = 3$, whence $\omega = 1$. So if ω is a root distinct from 1, S cannot be 3 and the remainder, $1 + \omega + \omega^2$, must be zero.

Are there just three roots? The division of $v^3 - 1 = 0$ by $v - 1$ and then by $v - \omega$, gave $v + 1 + \omega = 0$, with no remainder. If to this last equation we add ω^2 on both sides, we get $v + 0 = \omega^2$. Hence there are not *four* roots: the only roots of $v^3 = 1$ are 1, ω, and ω^2. But are there *three* roots, or may ω and ω^2 be the same root? To show that ω and ω^2 are distinct, consider the square of their difference. We have $(\omega - \omega^2)^2 = \omega^2 - 2\omega^3 + \omega^4 = \omega^2 - 2 + \omega = \omega^2 + 1 + \omega - 3 = 0 - 3$. Hence $\omega \neq \omega^2$. But we have not yet shown that either ω or ω^2 is distinct

Is there more than one root? So far we have *assumed* that $\omega \neq 1$, but now we can show that this inequality is valid. For we have the two equations

$$\omega - \omega^2 = \pm i\sqrt{3}$$
$$1 + \omega + \omega^2 = 0$$

Adding and subtracting, we get

$$1 + 2\omega = \pm i\sqrt{3}, \text{ whence } \omega = \tfrac{1}{2}(-1 \pm i\sqrt{3})$$
$$1 + 2\omega^2 = \mp i\sqrt{3}, \text{ whence } \omega^2 = \tfrac{1}{2}(-1 \mp i\sqrt{3})$$

Comparing these, we see that although there appear to be two values of ω, it is a matter of indifference which is taken, the other being then ω^2.

The mutual squares. Not only is ω^2 the square of ω, but also ω is the square of ω^2. For $(\omega^2)^2 = \omega^4 = (\omega^3)\omega = (1)\omega = \omega$. This curious property, that each is the square of the other is not possessed by any other pair of distinct numbers. For any pair of numbers, m and n, each of which is the square of the other, satisfy the equations $m^2 = n$ and $n^2 = m$, and so n must satisfy the equation $n^4 - n = 0$. Obvious factors are $(n - 0)$, $(n - 1)$, and $(n^2 + n + 1)$. This third factor may be written $n^2 - (\omega + \omega^2)n + \omega^3$, whose factors are obviously $(n - \omega)$ and $(n - \omega^2)$. Hence the only solutions are $n = 0, 1, \omega, \omega^2$, with

$m = 0, 1, \omega^2, \omega$. The pairs (0,0) and (1,1) are extraneous for our purpose, and (ω,ω^2) is the only pair of distinct numbers having the mutual square property.

Tufts College William R. Ransom

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

133. *Proposed by W. R. Talbot, Jefferson City, Missouri.*

If a, b, c and d are used to replace distinct non-zero digits, find their values in the equations

$$(ca)^2 + (ab)^2 = (cb)^2 + (c^2)^2 = (cc)^2 + (d)^2 = (bd)^2 + (bc)^2.$$

134. *Proposed by G. W. Courter, Baton Rouge, Louisiana.*

Using the sides of a parallelogram as hypotenuses, isosceles right triangles are constructed externally (or internally) to the parallelogram. Show that the vertices of the right angles determine a square.

135. *Proposed by C. S. Ogilvy, Syracuse University.*

A farmer sells p/q of his eggs plus p/q of an egg to his first customer, p/q of the remaining eggs plus p/q of an egg to his second customer, and so on until all of his eggs have been sold to n customers. Determine necessary and sufficient restrictions on p and q and find the initial number of eggs, if none are to be broken.

136. *Proposed by Corporal P. B. Beilin, Somewhere in Korea.*

What is the maximum number of spheres of radius r which can be placed in a cylindrical can of radius R and height H? (Thought of while eating canned pretzel balls.)

137. *Proposed by W. T. Cleagh, Jacksonville, Florida.*

Let $N = |\Pi p_i^{\alpha_k} \pm \Pi p_j^{\alpha_m}|$ where the sets p_i and p_j together constitute the first n primes and the α_k and α_m are arbitrary positive integers. Show that N is a prime if N is less than the square of the $(n + 1)$th prime. For example: $(2)^2(3)(5)(7)(11) - (13)(17)(19) = 421 < (23)^2$, so 421 is a prime.

138. *Proposed by D. Arany, Budapest, Hungary.*

Establish the following identity:
$[(AP)^2 - (AH)^2]\tan A + [(BP)^2 - (BH)^2]\tan B + [(CP)^2 - (CH)^2]\tan C = (PH)^2(\tan A + \tan B + \tan C)$, where P is an arbitrarily chosen point in the plane of the triangle ABC and H is the orthocenter of ABC.

139. *Proposed by H. J. Hamilton, Pomona College.*

Given a closed, convex curve C, not intersected by the x-axis. Let A be the area which C bounds and V the volume of the solid of revolution obtained by revolving A about the x-axis. Now V is given by each of two integral formulas, one obtained by the "circular disc method" of subdividing V and the other by the "cylindrical shell method." (See any elementary calculus text.) Reconcile these integrals without appealing directly to the concept of volume.

SOLUTIONS

Late Solution

105. *M. S. Klamkin, Polytechnic Institute of Brooklyn, New York.*

The Sum of Quadratic Surds

74. [Sept. 1950] *Proposed by Samuel Skolnik, Los Angeles City College.*

Prove that the sum of any finite number of dissimilar pure quadratic surds is irrational.

Solution by the Proposer. We have to prove that $\sum_{i=1}^{n} b_i\sqrt{a_i}$ is irrational, where $b_i \neq 0$, $a_i \neq 0$, and a_i contains no square factor.

Assume the proposition to be true for $n = k$. (It is well-known, or may easily be shown, that the proposition holds for $n = 2$.)

Now assume that $\sum_{i=1}^{k+1} b_i\sqrt{a_i} = r$, where r is rational. Then

$$\sum_{i=1}^{k} b_i\sqrt{a_i} - r = -b_{k+1}\sqrt{a_{k+1}}. \tag{1}$$

Consider the product of all expressions obtainable from $[x - (\sum_{i=1}^{k} b_i\sqrt{a_i} - r)]$ by keeping the signs of x and r unchanged and taking every possible arrangement of signs for the terms of $\Sigma b_i\sqrt{a_i}$. In the resulting polynomial, let c_i be the coefficient of $\sqrt{a_i}$. Since the polynomial is unchanged if $\sqrt{a_i}$ is replaced by $-\sqrt{a_i}$, we have $c_i\sqrt{a_i} = -c_i\sqrt{a_i}$ or $2c_i\sqrt{a_i} = 0$. But $a_i \neq 0$, so $c_i = 0$, and the product is a polynomial in x of degree 2^k with rational coefficients.

By (1), $-b_{k+1}\sqrt{a_{k+1}}$ is also a root of the polynomial, hence $b_{k+1}\sqrt{a_{k+1}}$ is also a root of the polynomial. This implies that for some two different arrangements of signs, the corresponding values of $\sum_{i=1}^{k} b_i\sqrt{a_i} - r$ are equal and opposite in signs. Then we have

$$\left(\sum_{i=1}^{k} \bar{b}_i\sqrt{a_i} - r\right) + \left(\sum_{i=1}^{k} \bar{\bar{b}}_i\sqrt{a_i} - r\right) = 0,$$

where $|\bar{b}_i| = |\bar{\bar{b}}_i| = |b_i|$. This implies

$$\sum_{i=1}^{k} (\bar{b}_i + \bar{\bar{b}}_i)\sqrt{a_i} = 2r, \tag{2}$$

where the number of non-vanishing terms $(\bar{b}_i + \bar{\bar{b}}_i)\sqrt{a_i}$ is certainly not greater than k.

Case 1. If $|r| > 0$, then (2) contradicts the induction hypothesis.

Case 2. If $r = 0$, then $\sum_{i=1}^{k+1} b_i\sqrt{a_i} = 0$ and $(\sum_{i=1}^{k+1} b_i\sqrt{a_i})^2 = 0$, whence

$$\sum_{i=1}^{k+1} b_i{}^2 a_i + \sum_{e=1}^{k+1}\sum_{m=1}^{k+1} 2b_e b_m\sqrt{a_e a_m} = 0, \tag{3}$$

where the terms such that $e = m$ are excluded from double summation. Now $\sum_{i=1}^{k+1} b_i^2 a_i > 0$, since it is a sum of non-vanishing positive numbers, so $\sum_{e=1}^{k+1}\sum_{m=1}^{k+1} 2b_e b_m\sqrt{a_2 a_m} \neq 0$. Moreover each $\sqrt{a_e a_m}$ is a quadratic irrational, since $a_e \neq a_m$ and neither contains a square factor. Therefore (3) is impossible by Case 1.

Since the assumption that the induction cannot be continued beyond $n = k$ has been shown to contradict the induction hypothesis itself, then the sum of any finite number of dissimilar pure quadratic surds is irrational.

A Conic Unrolled

87. [Jan. 1951] *Proposed by Leo Moser, Texas Technological College.*

A right circular cone is cut by a plane. The intersection of course is a conic. Find the equation of the curve that this conic goes into if the cone is unrolled on to a plane. In particular, if the cone is a cylinder and the plane cuts the axis of the cylinder at 45°, then the ellipse formed will unroll into a sine curve.

Solution by M. S. Klamkin's Sophomore Calculus Class, Polytechnic Institute of Brooklyn. Let the equation of the cone in cylindrical coordinates be $a^2r^2 = z^2$. Cut the cone along its intersection with the plane $y = 0$ and let that line become the x'-axis. Then the coordinates of the transform of a point (r, θ, z) on the cone are (r', θ'), where

$$r' = \sqrt{r^2 + z^2} = r\sqrt{1 + a^2}, \text{ and } \theta' = r\theta/\sqrt{r^2 + z^2} = \theta/\sqrt{1 + a^2}.$$

Now consider the transform of the intersection of the cone with a general surface, $F(r, \theta, z) = 0$. The equation of a cylinder with elements passing through the intersection curve and parallel to the z-axis is $F(r, \theta, ar) = 0$. Thus the equation of the transform curve will be

$$F(r'/\sqrt{1 + a^2},\ \theta'\sqrt{1 + a^2},\ ar'/\sqrt{1 + a^2}) = 0.$$

If $F(r, \theta, z) = 0$ is a plane, then

$$r(A \cos \theta + B \sin \theta) + Cz + D = 0$$

and the transform curve is

$$(r'/\sqrt{1 + a^2})(A \cos \theta'\sqrt{1 + a^2} + B \sin \theta'\sqrt{1 + a^2} + Ca) + D = 0.$$

If we use a cylinder, $r = a$, instead of the cone, $a^2r^2 = z^2$, we find that the point (r, θ, z) transforms into (x', y') where $x' = z$, and $y' = a\theta$. Thus if the curve of intersection is given by $r = a$ and $F(r, \theta, z) = 0$, then upon development the intersection is transformed into $F(a, y'/a, x') = 0$. Now if the intersecting surface is the plane $r(A \cos \theta + B \sin \theta) + Cz + D = 0$, then

$$F(a, y'/a, x') \equiv a(A \cos y'/a + B \sin y'/a) + Cx' + D = 0,$$

which is a sine curve for all plane intersections except when $A = B = 0$ or when $C = 0$.

Quadrisection of a Triangle

90. [Jan. 1951] *Proposed by D. L. MacKay, Manchester Depot, Vt.*

Triangle ABC is divided into two parts, triangle DBE and quadrilateral $ADEC$, by the line DE. Construct a line which will bisect each of these parts.

Solution by the Proposer. Let PM cut DE in N with P on AB and M on AC. Now the envelope of a line PN which bisects a given triangle DBE is a hyperbola, for setting $DN = x$, $DP = y$, we have $xy = \frac{1}{2}(DB)(DE)$ = a constant. The center of the hyperbola is D and its asymptotes are the indefinite sides DB and DE. Corresponding to the vertices B and E we have two other hyperbolas. As P and N traverse the perimeter so that PN bisects triangle DBE, the tangency of PN

passes from one to another of these hyperbolas when PN coincides with one of the medians. [See this MAGAZINE, **24**, 167, (Jan. 1951).]

Prolong ED to cut CA extended at O and set $OM = x'$, $ON = y'$. Then triangle OMN/triangle $OEC = x'y'/(OE)(OC) = k$, a constant. Hence $x'y' = k(OE)(OC)$ = a constant and the envelope of NM is a hyperbola having O for center and OE and OC as asymptotes.

Now PM is the common tangent to these two hyperbolas. In general the construction of this tangent would involve a fourth degree equation, but since the two hyperbolas have a common asymptote, OE, the equation is reduced to a quadratic equation.

Let $AD = a$, $AO = b$, $DO = c$, $AB = d$, $AC = e$, $DE = f$, and $AP = x$. Then $DP = x - a$ and $DB = d - a$. Since $(AM)(AP)/(AB)(AC)$ = triangle APM/triangle $ABC = 1/2$, we have $AM = de/2x$. Also, $(DP)(DN)/(DB)(DE)$ = triangle DPN/triangle $DBE = 1/2$, so $DN = (d - a)f/2(x - a)$.

Now draw PH parallel to ED and cutting CA extended at H. Then from $AD/AP = AO/AH = DO/PH$ we obtain $AH = bx/a$ and $PH = cx/a$. From $MH/MO = PH/NO$, we have

$$NO = [(cx/a)(b + de/2x)]/[bx/a + de/2x] = (2bcx^2 + cdex)/(2bx^2 + ade).$$

From $DN = NO - DO$, we have

$$(d - a)f/2(x - a) = (cdex - adec)/(2bx^2 + ade), \text{ or}$$

$$2[bf(d - a)/de - c]x^2 + 4acx + a[df - af - 2ac] = 0.$$

We obtain AP by constructing the positive root of this equation and AM by the construction of $de/2x$.

According to D. J. Korteweg, *Complete Works of Huygens*, **11**, 219-225, this problem received the attention of Huygens in 1650, 1656 and 1659.

If in the problem we replace the given line DE by the condition that triangle ABC be divided into four equal parts by two perpendicular lines, we have a problem sometimes called the Problem of Leibniz, since it is mentioned in one of his works, *Nova Algebrae promotio* – Gerhard's ed. Leibniz, *Mathematische Schriften*, **7**, Halle, 154, (1863). It has appeared as an exercise in seven editions of *Traité de Geométrie* of Rouché et de Comberousse. It has evoked many sterile "solutions," for the problem cannot be constructed by straight edge and compass. – *L'Intermédiaire des Mathematiciens*, **1**, 39, 55-62, 135, (1894).

A Curve Dividing a Rectangle

110. [Sept. 1951] *Proposed by H. T. R. Aude, Colgate University.*

(a) Place a unit square with its sides parallel to the coordinate axes so that one curve of the family $x^2y = c$ will pass through two opposite corners and divide the area of the square in the ratio 1:3.

(b) Consider the similar problem when the square is replaced by a rectangle a by b and the ratio of the division of its area is the proper fraction $n:m$.

Solution by Lt. Col. R. E. Horton, Lackland AFB, Texas. In (b) let the vertices of the rectangle be (x_1, y_1), (x_2, y_1), (x_2, y_2), and (x_1, y_2) where

$$x_2 - x_1 = a \quad \text{and} \quad y_1 - y_2 = b. \tag{1}$$

Also let the desired curve of the family be $x^2y = c_1$, whereupon

$$y_1 = c_1/x_1^2 \quad \text{and} \quad y_2 = c_1/x_2^2. \tag{2}$$

Then we have

$$m\int_{x_1}^{x_2}(c_1/x^2 - c_1/x_2^2)dx = n\int_{x_1}^{x_2}(c_1/x_1^2 - c_1/x^2)dx.$$

Upon integrating and simplifying we get $(x_2 - x_1)^2(nx_2 - mx_1) = 0$ which leads to the trivial solution $x_2 = x_1$ and one other, $x_1 = nx_2/m$. With this and equations (1) and (2) we arrive at the solution: $x_1 = an/(m - n)$; $y_1 = bm^2/(m^2 - n^2)$; $x_2 = am/(m-n)$; $y_2 = bn^2/(m^2 - n^2)$; $c_1 = a^2bm^2n^2/(m^2 - n^2)(m - n)^2$. Thus the points (x_1, y_1), (x_2, y_2) and the curve $x^2y = a^2bm^2n^2/(m + n)(m - n)^3$ satisfy the conditions of part (b).

(a) When $a = b = 1 = n$ and $m = 3$, we have $(x_1, y_1) = (1/2, 9/8)$, $(x_2, y_2) = (3/2, 1/8)$ and $x^2y = 9/32$ is the curve through the two points.

Also solved by *W. B. Carver, Cornell University; A. Sisk, Maryville, Tenn.; and the proposer.*

The Range of a Projectile

111. [Sept. 1951] *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D.C.*

A projectile is fired at an angle of elevation θ and with initial velocity u. After a time t_1 the projectile is at a point P where it suddenly receives an added velocity v directed along the tangent to the trajectory at P. Find an expression for the range of the projectile in terms of θ, u, t_1, and v. (Consider gravity as the only force acting.)

Solution by Howard Eves, Champlain College. Let R be the range, R_1 and R_2 the horizontal distances travelled by the projectile, in times t_1 and t_2, before and after P. Taking the origin at the initial point we have the well-known parametric representation for the first part of the trajectory:

$$x = ut \cos \theta, \quad y = ut \sin \theta - \tfrac{1}{2}gt^2. \tag{1}$$

Let w be the tangential velocity at P (v not yet added). Then from (1) we find

$$w = [u^2 - 2gut_1 \sin \theta + g^2 t_1^2]^{1/2}. \tag{2}$$

Also, if ϕ is the inclination of vector w, we have

$$\cos \phi = u \cos \theta / w, \quad \sin \phi = (u \sin \theta - gt_1)/w. \tag{3}$$

Let h be the height of the projectile at P. Then

$$h = ut_1 \sin \theta - \tfrac{1}{2}gt_1^2. \tag{4}$$

We can now calculate t_2 (assuming the case where P is before the maximum height) as

$$t_2 = \{(v + w) \sin \phi + [(v + w)^2 \sin^2\phi + 2gh]^{1/2}\}/g. \tag{5}$$

But

$$R = R_1 + R_2 = ut_1 \cos \theta + (v + w)t_2 \cos \phi.$$

Substituting from (2), (3), (4), (5) we obtain the desired relation.

This result furnishes a first approximation for problems connected with rocket bombs.

Also solved by *Leon Bankoff, Los Angeles, California; Louis Berkofsky, Roxbury, Massachusetts; W. B. Carver, Cornell University; and the proposer.*

A Nine Digit Square

112. [Nov. 1951] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Find a number of the form *aaabbbccc* which gives, when increased by unity, a perfect square of nine digits.

Solution by T. W. Carlos, Detroit, Michigan. If N^2 is to terminate in ccd, where $d = c + 1$, then N has one of the forms $100k$, $250k \pm 1$, $250k \pm 83$, or $500k \pm 166$. If N^2 begins with aaa, then N must fall within definite ranges which may be selected from a table of the squares of four-digit integers, for example, 10530 to 10590, 14890 to 14940, etc. There are only five values of N of one of the necessary forms within these ranges, namely: 21083, 18251, 18249, 14917, and 10583.

$aaabbbccc = (10^6 a + 10^3 b + c)(111) = (N + 1)(N - 1)$. Hence one of the factors $(N + 1)$ or $(N - 1)$ is divisible by 37, and one of the factors is divisible by 3. The test for divisibility by 37 eliminates the first four of the possible values of N, so that the unique solution is

$$111999888 = (10583)^2 - 1.$$

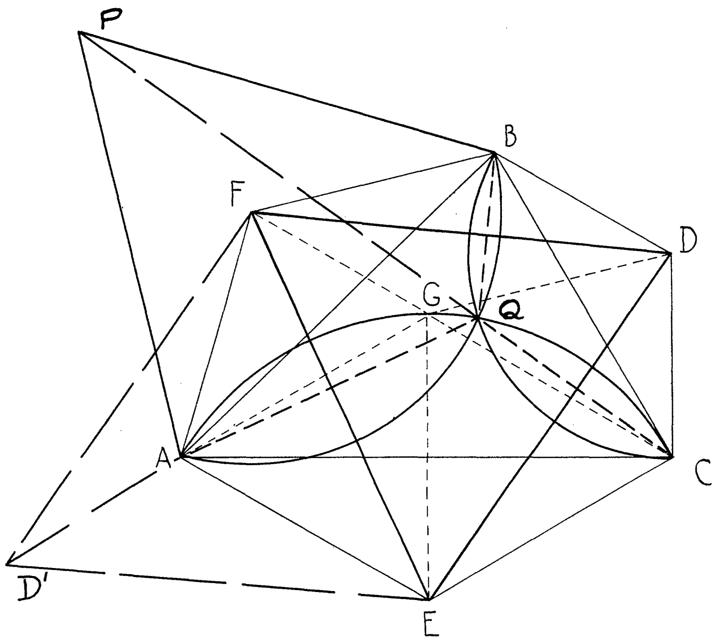
Also solved by *Leon Bankoff, Los Angeles, Calif.; Monte Dernham, San Francisco, Calif.; Erich Michalup, Caracas, Venezuela; F. L. Miksa, Aurora, Ill.; and J. S. Shipman, Laboratory for Electronics, Inc., Boston, Mass.*

Michalup points out that if zero be considered an admissible value for a, then $000444888 = (667)^2 - 1$ and $000111555 = (334)^2 - 1$ are also solutions.

Equilateral Triangle in Isogonic Configuration

113. [Nov. 1951] *Proposed by Benjamin Greenberg, Brooklyn, N.Y.*

Isosceles triangles with base angles of 30° are constructed externally on the sides of triangle ABC. The third vertices of the isosceles triangles determine an equilateral triangle. Can this be proven by pure synthetic geometry without recourse to trigonometry?



I. *Solution by Charles Salkind, Polytechnic Institute of Brooklyn.* Let the third vertices of the isosceles triangles on AB, BC, CA be F, D, E, respectively. On AE, with $D'A = DC$ and $D'E = DE$, construct triangle $D'AE$ congruent to triangle DCE. Then angle $D'AE$ = angle DCE = angle $BCA + 60°$. Now angle FAE = angle $BAC + 60°$, so angle $D'AF$ = $360°$ - (angle BCA + angle $BAC + 120°$) = $360°$ - ($180°$ - angle ABC + $120°$) = angle $ABC + 60°$ = angle FBD. Draw $D'F$. Then $D'A = DC = DB$ and $AF = FB$, so triangles $D'AF$ and DBF are congruent. Hence $D'F = DF$, and triangles FDE and $FD'E$ are congruent.

Angle DEC = angle $D'EA$ and angle DFB = angle $D'FA$, so angle $D'ED$ = angle $AEC = 120°$ = angle AFB = angle $D'FD$. Now since triangles $D'FE$ and DFE are congruent, angle $D'EF$ = angle DEF = ½ angle $D'ED$ = $60°$, and angle $D'FE$ = angle DFE = ½ angle $D'FD = 60°$. Hence angle $EDF = 180°$ - angle DEF - angle $DFE = 60°$ and triangle DEF is equilateral.

II. *Solution by Leon Bankoff, Los Angeles, California.* Let D, E, F denote the vertices of the externally constructed isosceles triangles opposite A, B, C, respectively. On FB and BD construct a parallelogram with fourth vertex at G. Draw GA, GE and GC. Then angle BDG = angle GFB. Now angle $BDC = 120°$ = angle AFB, hence angle GDC = angle AFG. Also, $GD = FB = AF$ and $DC = BD = FG$, so triangles GDC and AFG are congruent. Hence $GC = AG$ and angle CGD = angle GAF.

In parallelogram $FBDG$, angle $FGD = 180°$ - angle BFG
= $180°$ - ($120°$ - angle AFG) = $60°$ + angle AFG. Then angle AGC
= $360°$ - (angle FGA + angle FGD + angle CGD)
= $360°$ - (angle FGA + $60°$ + angle AFG + angle GAF)
= $360°$ - ($60°$ + $180°$) = $120°$.

$GC = AG$, $AE = EC$, and $GE = GE$. Therefore triangles AGE and CGE are congruent. It follows that angle EGC = angle AGE = ½ angle AGC = $60°$ and angle GEC = angle GEA = ½ angle $AEC = 60°$. Then angle GCE = $60°$ = angle GAE and triangles AGE and CGE are equilateral. Hence $GE = CE$ and angle GCE = angle AGE. From triangles GDC and AFG, angle DCG = angle FGA, so angle DCE = angle FGE. Therefore, since $DC = FG$, triangles DCE and FGE are congruent and $DE = FE$.

In like manner it may be shown that $FE = FD$, whereupon triangle FDE is equilateral.

III. *Solution by F. F. Dorsey, South Orange, N.J.* On any triangle ABC construct externally isosceles triangles ABF, BCD, and CAE with base angles of $30°$. Also, on AB construct an exterior equilateral triangle ABP and draw CP. Triangles DBF and CBP are similar, for $BC/BD = BP/BF$ and angle DBF = angle $ABC + 60°$ = angle CBP. Therefore $FD/CP = BF/BP$. In like manner, triangles EAF and CAP are proven similar, and $FE/CP = AF/AP = BF/BP = FD/CP$, whence $FE = FD$. By the same method, it may be shown that $DE = FD$. Hence EFD is equilateral.

IV. *Solution by W. B. Carver, Cornell University.* Let A, B, C be the vertices of the given triangle in clockwise order and D, B, C; A, E, C; A, B, F, the vertices of the isosceles triangles in counterclockwise order. Note that F is the center of an equilateral triangle constructed externally on AB. Draw the $120°$ arc AB of the circle with center at F, and with D as center draw the similar arc BC intersecting the arc AB at Q. Draw the line segments QA, QB, QC. Angle $AQB = 120° =$ angle BQC, hence angle $CQA = 120°$ and Q lies also on the $120°$ arc AC with center at E. (This point Q is one of the "isogonic centers." See R. A. Johnson, *Modern Geometry*, page 218.) Since F and D are both equidistant from B and Q, line FD is perpendicular to BQ; and similarly lines DE and EF are respectively perpendicular to lines CQ and AQ. It follows that angle $FDE = 60° =$ angle DEF $=$ angle EFD, whereupon triangle EFD is equilateral.

The above proof is for the case where all the angles of the triangle ABC are less than $120°$. The proof has to be slightly modified in an obvious way for the case of a triangle with one angle greater than or equal to $120°$.

Also solved by *Charles Salkind*, using methods I, III and IV.

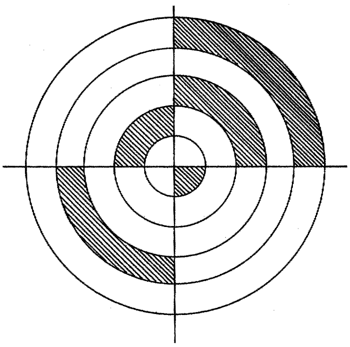
QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 61. Multiply 5746320819 by 125. [*Submitted by W. C. True.*]

Q 62. Show that if an even number is multiplied by 6, the unit's digits of the even number and of the product will be the same. If an odd number is multiplied by 6, the unit's digits will differ by 5. [*Submitted by W. R. Ransom.*]

Q 63. The figure consists of five concentric circles of radii 1, 2, 3, 4, and 5 inches. The two diameters are mutually perpendicular. Find



the total shaded area. [*Submitted by the Department of Mathematics, Woodrow Wilson Junior College, Chicago, Ill.*]

Q 64. Solve the system: $xy/(x + y) = a$, $xz/(x + z) = b$, $yz/(y + z) = c$. [*From the 1951 High School Mathematical Contest of the Metropolitan New York Section of the Mathematical Association of America. By permission.*]

Q 65. Find the sum of the coefficients of the expansion of $(x + y)^n$. [*Submitted by T. E. Sydnor.*]

Q 66. The axes of symmetry of two 2″ right circular cylinders intersect at right angles. What volume do the cylinders have in common? [*Submitted by G. R. Jaffray.*]

ANSWERS

A 61. Since $125 = 100/8$, then $5746320819000/8 = 718290102375$.

A 62. Since $(2n \times 6) - 2n = 10n$, then $2n \times 6$ and $2n$ must have the same unit's digit. Since $(2n + 1) \times 6 - (2n + 1) = 10n + 5$, the unit's digits differ by 5. (This "discovery" was taken from the Diary of a 13 year old boy.)

A 63. Rotate each of the inner rings until all the shaded areas fall in the same quadrant. Then the shaded area is $\pi(5)^2/4 \doteq 19.63$ square inches. The area is determined by the outer radius and is independent of the inner radii.

A 64. $1/y + 1/x = 1/a$, $1/z + 1/x = 1/b$, and $1/z + 1/y = 1/c$. From the sum of the first two equations subtract the third, and obtain $2/x = 1/a + 1/b - 1/c$. Hence $x = 2abc/(bc + ac - ab)$. It follows from symmetry that $y = 2abc/(ab + bc - ac)$ and $z = 2abc/(ac + ab - bc)$.

A 65. In the expansion, $(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + y^n$, let $x = y = 1$. Then $2^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + 1$.

A 66. $2(0 + 4 \cdot 4 + 0)/6$ or $16/3$ cubic inches, since the prismoidal formula gives exact values of an integral when the integrand is a polynomial of the third degree or less.

Second method by *Leo Moser*. Consider any cross section of the volume parallel to the square mid section. This will be a square whose inscribed circle will be the cross section of the inscribed sphere of the common volume. Hence the common volume is to the volume of the sphere as the area of the square is to the area of its inscribed circle. Then $V = (4\pi r^3/3)(4/\pi) = 16r^3/3$.

OUR CONTRIBUTORS

(*Continued from back of Table of Contents.*)

1947 to 1951. He has published research on the theory of Banach spaces, linear topological spaces, and topological groups. He is one of the editors of the "Mathematics Dictionary".

John W. Green, Associate Professor of Mathematics, University of California, Los Angeles, was born in Hearn, Texas in 1914. A graduate of the Rice Institute (B.A. '35, M.A. '36) and of the University of California (Ph.D. '38), Dr. Green taught at Harvard University and the University of Rochester and in 1943 became a mathematician at Aberdeen Proving Ground. He was appointed to the faculty of U.C.L.A. in 1945. He has served as associate editor, Duke Mathematical Journal, and as an Associate Secretary, American Mathematical Society. During the current year Professor Green is at the Institute of Advanced Study, Princeton, on sabbatical leave. His article on the mathematics of ballistics appeared in the November-December issue.

John W. Odle was born in Tipton, Indiana, in 1914. After obtaining his collegiate training at the University of Michigan (B.S. 1937, M.S. 1938, Ph.D. 1940) he was instructor of mathematics for two years at the University of Wisconsin. He then went to Pennsylvania State College as an assistant professor, and in 1944 he joined the 8th Air Force in England as a civilian operations analyst. In 1946 Dr. Odle shifted his allegiance to the Navy and became a research mathematician at the Naval Ordnance Test Station, China Lake, California. He is now head of the mathematics division there. His article on mathematical careers in military research came out in the January-February number.

Alexander J. Yannopoulos, Professor of Mathematics, Greek Aviation Academy and Assistant Professor, University of Athens, was born in Eleusis, Attiki, Greece in 1916. He received his doctor's degree at the University of Athens in 1950. A member of the Committee of Directors of the Mathematical Society of Greece, he is interested especially in differential geometry, calculus of variations and the mathematics of insurance. His article on a general trihedron of a curve was published in the March-April issue.